It is evident that W^μ is translationally invariant, $[P_\mu, W_\nu] = 0$. W^2 is a Lorentz scalar, $[J_{\mu\nu}, W^2]$, as you will explicitly show in homework. Here $\epsilon^{\mu\nu\lambda\sigma}$ is the totally antisymmetric invariant tensor,

$$\begin{aligned}\overline{\epsilon}^{\mu\nu\lambda\sigma} &= (g^{\mu\mu'} - \delta\omega^{\mu\mu'})(g^{\nu\nu'} - \delta\omega^{\nu\nu'})(g^{\lambda\lambda'} - \delta\omega^{\lambda\lambda'})(g^{\sigma\sigma'} - \delta\omega^{\sigma\sigma'})\epsilon_{\mu'\nu'\lambda'\sigma'} \\ &= \epsilon^{\mu\nu\lambda\sigma}, \end{aligned} \tag{1.167}$$

(because $\epsilon^{\mu\nu\lambda\sigma}$ vanishes if any two indices are the same), where $\epsilon^{0123} = +1$. In the rest frame of a particle,

$$\mathbf{P} = \mathbf{0}, \quad P^0 = E = m, \tag{1.168}$$

and so

$$W^0 = \frac{1}{2}\epsilon^{0ijk} J_{ij} P_k = 0, \tag{1.169a}$$

$$W^i = -\frac{1}{2}\epsilon^{ijk0} J_{ij} m = m\frac{1}{2}\epsilon^{ijk} J_{jk} = mJ^i, \tag{1.169b}$$

where the latter is the spin. Thus, the eigenvalues of W^2 are

$$W^2 = m^2 s(s+1). \tag{1.170}$$

This means for a particle with nonzero rest mass, $m^2 > 0$, the irreducible representations belong to the values $s = 0, 1/2, 1, \ldots$. For a given s, the possible value of J_3 are $s_3 = -s, -s+1, -s+2, \ldots, s-1, s$. The massless limit has to be taken carefully (see homework):

$$m = 0: \quad W^\mu = \lambda P^\mu, \quad \lambda = \frac{\mathbf{P}\cdot\mathbf{S}}{P^0}. \tag{1.171}$$

λ is called the helicity, which is the spin projected along the direction of motion.

There are other representations of the Poincaré group, such as tachyons, where $m^2 < 0$, but they seem *not* to be realized in nature.

1.7 Plane-wave Solutions of the Dirac Equation

If $\psi = e^{ipx} u_p$, where $px = p^\mu x_\mu = \mathbf{p}\cdot\mathbf{x} - Et$, the Dirac equation becomes

$$(\gamma p + m) u_p = 0. \tag{1.172}$$

For a particle at rest, $p^0 = m$, $\mathbf{p} = \mathbf{0}$, this is

$$(1 - \gamma^0) v = 0, \tag{1.173}$$

where $v = u_{\mathbf{p}=\mathbf{0}}$ is the rest-frame spinor. This means that v is an eigenvector of γ^0 with eigenvalue $+1$. Because

$$[\Sigma_3, \gamma^0] = 0, \tag{1.174}$$

we can also take v to be an eigenvector of Σ_3:

$$\Sigma_3 v_\sigma = \sigma v_\sigma, \quad \sigma = \pm 1. \tag{1.175}$$

We can obtain $u_{p\sigma}$ from the rest-frame spinor v_σ by a boost:

$$\begin{aligned} u_{p\sigma} &= \exp\left[\phi\,\mathbf{e}\cdot\frac{1}{2}\boldsymbol{\alpha}\right] v_\sigma \\ &= \left(\cosh\frac{1}{2}\phi + \boldsymbol{\alpha}\cdot\mathbf{e}\sinh\frac{1}{2}\phi\right) v_\sigma, \end{aligned} \tag{1.176}$$

because $(\boldsymbol{\alpha}\cdot\mathbf{e})^2 = 1$. Here, the direction of the boost is given by that of the momentum,

$$\mathbf{e} = \frac{\mathbf{p}}{|\mathbf{p}|}. \tag{1.177}$$

Because

$$\cosh\phi = \frac{1}{\sqrt{1-v^2}} = \frac{E}{m}, \tag{1.178}$$

we have

$$\cosh\frac{\phi}{2} = \sqrt{\frac{\cosh\phi + 1}{2}} = \sqrt{\frac{E+m}{2m}}, \tag{1.179a}$$

$$\sinh\frac{\phi}{2} = \sqrt{\frac{\cosh\phi - 1}{2}} = \sqrt{\frac{E-m}{2m}}, \tag{1.179b}$$

and therefore

$$\begin{aligned} u_{p\sigma} &= \left(\sqrt{\frac{E+m}{2m}} + \frac{\boldsymbol{\alpha}\cdot\mathbf{p}}{\sqrt{E^2-m^2}}\sqrt{\frac{E-m}{2m}}\right) v_\sigma \\ &= \frac{1}{\sqrt{2m(E+m)}}(E+m+\boldsymbol{\alpha}\cdot\mathbf{p})v_\sigma \\ &= \frac{1}{\sqrt{2m(E+m)}}(m+\gamma^0 E - \boldsymbol{\gamma}\cdot\mathbf{p})v_\sigma \\ &= \frac{1}{\sqrt{2m(E+m)}}(m-\gamma p)v_\sigma. \end{aligned} \tag{1.180}$$

This evidently satisfies the Dirac equation (1.172):

$$(m+\gamma p)u_{p\sigma} \propto (m+\gamma p)(m-\gamma p)v_\sigma = (m^2+p^2)v_\sigma = 0, \tag{1.181}$$

because $(\gamma p)^2 = -p^2$. We also note that since $\{v_\sigma\}$, $\sigma = \pm 1$, span the two-dimensional space for which $\gamma^0 = 1$, we must have the projection-operator statement

$$\frac{1}{2}(1+\gamma^0) = \sum_\sigma v_\sigma v_\sigma^\dagger. \tag{1.182}$$

If we multiply both sides of this equation by $v_{\sigma'}$,

$$v_{\sigma'} = \sum_\sigma v_\sigma (v_\sigma^\dagger v_{\sigma'}), \tag{1.183}$$

which implies that the rest-frame spinors are orthonormal,

$$v_\sigma^\dagger v_{\sigma'} = \delta_{\sigma\sigma'}. \tag{1.184}$$

A Lorentz-invariant way of writing these two results, which you will explicitly prove in Homework, is

$$\frac{m - \gamma p}{2m} = \sum_\sigma u_{p\sigma} u_{p\sigma}^\dagger \gamma^0, \tag{1.185a}$$

$$u_{p\sigma}^\dagger \gamma^0 \gamma^\mu u_{p\sigma'} = \delta_{\sigma\sigma'} \frac{p^\mu}{m}. \tag{1.185b}$$

We also note that v_σ^* is an eigenvector of γ^0 with eigenvalue -1:

$$\gamma^0 v_\sigma^* = -v_\sigma^*, \quad \Sigma_3 v_\sigma^* = -v_\sigma^*, \tag{1.186}$$

because γ^0 and Σ_3 are imaginary. The corresponding projection operator statement is

$$\frac{1}{2}(1 - \gamma^0) = \sum_\sigma v_\sigma^* v_\sigma^T. \tag{1.187}$$

These spinors correspond to *negative-energy* solutions,

$$u_{p\sigma}^* = \frac{1}{\sqrt{2m(E+m)}}(m + \gamma p) v_\sigma^*, \tag{1.188}$$

which satisfies

$$(m - \gamma p) u_{p\sigma}^* = 0, \tag{1.189}$$

implying a plane wave solution of the form

$$\psi = e^{-ipx} u_{p\sigma}^*. \tag{1.190}$$

It is the appearance of these negative-energy solutions that destroys all hope of a wavefunction interpretation of ψ. (An example is given by the Klein paradox, see, e.g., Bjorken and Drell.) A partial resolution of the difficulty is the Dirac *hole* theory, in which all negative-energy states are filled. A vacancy (hole) in the sea of negative energy states appears as a positive-energy *antiparticle*; for electrons, the hole is a positron. However, we will not pursue this line of thought, for a more thoroughgoing reconstruction of the theory is necessary.

A final note. Recall $\gamma_5 = \gamma^0\gamma^1\gamma^2\gamma^3$ discussed earlier. Note that we can write

$$\Sigma_1 = \sigma_{23} = \frac{i}{2}[\gamma_2, \gamma_3] = i\gamma_2\gamma_3, \tag{1.191}$$

so

$$i\gamma_5 \Sigma_1 = i\gamma^0\gamma^1\gamma^2\gamma^3 i\gamma_2\gamma_3 = \gamma^0\gamma^1, \tag{1.192}$$

or generally,

$$\gamma^0 \boldsymbol{\gamma} = i\gamma_5 \boldsymbol{\Sigma}. \tag{1.193}$$

Then the boost equation (1.180) becomes

$$u_{p\sigma} = \frac{1}{\sqrt{2m(E+m)}} \left(E + m + i\gamma_5 \boldsymbol{\Sigma} \cdot \mathbf{p}\right) v_\sigma. \tag{1.194}$$

We can take $\mathbf{p}$ to be the quantization direction for $\boldsymbol{\Sigma}$:

$$\boldsymbol{\Sigma} \cdot \mathbf{p} = |\mathbf{p}|\sigma = \sqrt{E^2 - m^2}\,\sigma, \tag{1.195}$$

where now σ is the helicity. Further, from the Homework,

$$i\gamma_5 \sigma v_\sigma = v^*_{-\sigma}. \tag{1.196}$$

So then we can write

$$u_{p\sigma} = \frac{1}{\sqrt{2m}} \left(\sqrt{E+m}\, v_\sigma + \sqrt{E-m}\, v^*_{-\sigma}\right). \tag{1.197}$$

Left- and right-handed spinors are obtained by projecting with $\frac{1}{2}(1 \mp i\gamma_5)$:

$$u_{L,R} = \frac{1}{2}(1 \mp i\gamma_5) u. \tag{1.198}$$

Note that $i\gamma_5$ is a good quantum number, the chirality, if $m = 0$. Consider a general Lorentz transformation,

$$\begin{aligned} u_{L,R} &\rightarrow \frac{1}{2}(1 \mp i\gamma_5)\left(1 + i\boldsymbol{\delta\omega} \cdot \frac{1}{2}\boldsymbol{\Sigma} - \delta\mathbf{v} \cdot \frac{1}{2}\boldsymbol{\alpha}\right) u \\ &= \left(1 + i\boldsymbol{\delta\omega} \cdot \frac{1}{2}\boldsymbol{\Sigma} \pm \delta\mathbf{v} \cdot \frac{1}{2}\boldsymbol{\Sigma}\right) u_{L,R}, \end{aligned} \tag{1.199}$$

because $\boldsymbol{\alpha} = i\gamma_5 \boldsymbol{\Sigma}$. This indeed is the correct transformation properties for the $(1/2, 0)$ and $(0, 1/2)$ representations, respectively. See (1.123), (1.124).

1.8 Irreducible Representations of the Lorentz Group

Another way to describe Lorentz transformations is the following. Associated with any four-vector x^μ is a 2×2 Hermitian matrix,

$$x = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} = x^0 1 + \mathbf{x} \cdot \boldsymbol{\tau}, \tag{1.200}$$

where $\boldsymbol{\tau}$ are the Pauli matrices. The scalar length of x^μ is given by the determinant of this matrix:

$$x^\mu x_\mu = -(x^0)^2 + \mathbf{x} \cdot \mathbf{x} = -\det x. \tag{1.201}$$

We can extract x^μ from the matrix x as follows:

$$x^\mu = \frac{1}{2}\mathrm{Tr}\,(x\tau^\mu), \quad \tau^\mu = (1, \boldsymbol{\tau}). \tag{1.202}$$

If A is any matrix with determinant unity, $\det A = 1$, we can construct a new matrix $\hat{x}$ by the transformation

$$\hat{x} = AxA^\dagger, \tag{1.203}$$

which has the same determinant as x:

$$\det \hat{x} = \det x, \tag{1.204}$$

so the corresponding four-vector has the same length as that of x^μ: $\hat{x}^\mu \hat{x}_\mu = x^\mu x_\mu$. That is, A corresponds to a restricted Lorentz transformation,

$$SO(3,1) = SL(2,C), \tag{1.205}$$

the latter being the group of transformations induced by 2 complex matrices with determinant 1, a "special linear" group.

The irreducible representations of the Lorentz group are given by, as a generalization of the above transformation of a vector,

$$\xi_{\alpha_1\ldots\alpha_j;\dot{\beta}_1\ldots\dot{\beta}_k} \to A_{\alpha_1\rho_1}\cdots A_{\alpha_j\rho_j}A^*_{\dot{\beta}_1\dot{\sigma}_1}\cdots A^*_{\dot{\beta}_k\dot{\sigma}_k}\xi_{\rho_1\ldots\rho_j;\dot{\sigma}_1\ldots\dot{\sigma}_k}. \tag{1.206}$$

This belongs to the representation $(\frac{1}{2}j, \frac{1}{2}k)$, which is characterized by j undotted indices (which transform with A) and k dotted indices (which transform with $A^\dagger$). For more details on this way of proceeding see Gel'fand, Minlos, and Shapiro, *Representations of the Rotation and Lorentz Groups*, Pergamon Press, 1963.

Tensors are not, in general, irreducible representations. For example, consider a general second rank tensor, $A^{\mu\nu}$. It can be decomposed as follows:

$$A^{\mu\nu} = g^{\mu\nu}A + F^{\mu\nu} + T^{\mu\nu}, \tag{1.207}$$

where

$$T^{\mu\nu} = T^{\nu\mu}, \quad T^\mu{}_\mu = 0, \quad \text{(symmetric and traceless)} \tag{1.208}$$

and

$$F^{\mu\nu} = -F^{\nu\mu}, \quad \text{(antisymmetric)}. \tag{1.209}$$

The count of independent components is consistent:

$$16 = 1 + 6 + 9 = 1 + (3+3) + (3\times 3). \tag{1.210}$$

The latter count refers to the spinorial representation:

$$\begin{aligned}\left(\frac{1}{2},\frac{1}{2}\right)\otimes\left(\frac{1}{2},\frac{1}{2}\right) &= \left(\frac{1}{2}\otimes\frac{1}{2},\frac{1}{2}\otimes\frac{1}{2}\right)\\ &= (1,1)\oplus(1,0)\oplus(0,1)\oplus(0,0),\end{aligned} \tag{1.211}$$

where the first term corresponds to $T^{\mu\nu}$, the second and third to $F^{\mu\nu}$, and the last to A.