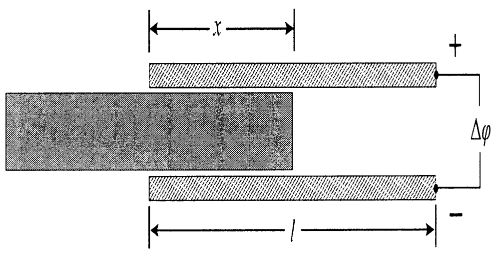


1. A parallel plate capacitor has the region between its plates filled with a dielectric slab of dielectric constant $K = \epsilon/\epsilon_0$ and mass m. The plate dimensions are: width w, length ℓ, and plate separation d. The capacitor plates are connected to a battery of constant voltage V ($\Delta\phi = V$ in the figure). Neglect the fringe field and friction, and assume the slab is constrained to move in the plane parallel to the capacitor plates.

   (a) {2 pts} Compute the capacitance $C \equiv q/V$ of this capacitor as a function of x.

   (b) {2 pts} If the slab is withdrawn half way (to $x = \ell/2$) and held in place, what is the magnitude and direction of the force on the slab caused by the electric field?

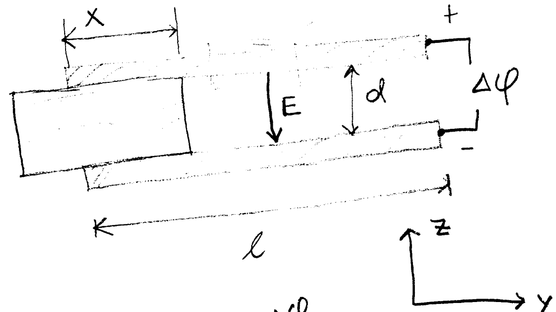
   (c) {2 pts} At $x = \ell/2$ the slab is released and given a velocity v_0 to the right. Find the current supplied by the battery at the instant it is released.

   (d) {2 pts} At $x = \ell/2$ the slab is again released but with zero velocity. Describe the motion of the slab (in words). What is the maximum velocity achieved by the slab?

   (e) {2 pts} Sketch the displacement of the slab versus time.

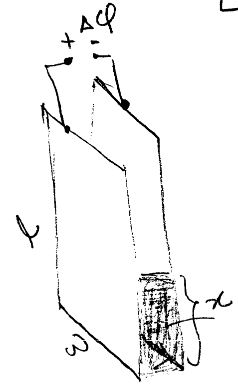
Problem 1 (Gaussian)

a) $$V = -\int_0^d \bar{E}\cdot d\bar{l}$$



$$= -\int_0^d E(-\hat{z})\cdot\hat{z}\,dz$$

$$= \int_0^d E\,dz = Ed$$



$$\Rightarrow \quad \bar{E} = \frac{V}{d}(-\hat{z})$$

So, using a Gaussian pillbox

$$\int \bar{E}\cdot d\bar{a} = 4\pi Q_{enc}$$

$$\Rightarrow \int_0^x \epsilon E w\,dx + \int_x^l E w\,dx = 4\pi Q$$

$$\Rightarrow +\frac{\epsilon V w}{d}x + \frac{Vw}{d}(l-x) = 4\pi Q$$

$$\Rightarrow \frac{Vw}{d}(\epsilon x + l - x) = 4\pi Q$$

$$\Rightarrow V = 4\pi\frac{Qd}{w\{l+(\epsilon-1)x\}}$$

$$C = \frac{Q}{V} = \frac{w\{l+(\epsilon-1)x\}}{4\pi d}$$

b) Assuming the charge Q on the capacitor held constant

the energy stored in the capacitor

$$W = \frac{1}{2}\frac{Q^2}{C}$$

Now, the electric force on the slab

$$F = -\frac{dW}{dx} = -\frac{1}{2}\frac{d}{dx}\left(\frac{Q^2}{C}\right)$$

$$\Rightarrow F = \frac{Q^2}{2C^2}\frac{dC}{dx}$$

where, $\frac{dC}{dx} = \frac{d}{dx}\left\{\frac{w(l+(\epsilon-1)x)}{4\pi d}\right\}$

$$= \frac{w(\epsilon-1)}{4\pi d}$$

$$F = \frac{Q^2}{2C^2}\frac{w(\epsilon-1)}{4\pi d} = \frac{V^2 w(\epsilon-1)}{8\pi d}$$

Since, the capacitor is connected with battery it is more logical to Keep the potential const but in that case we have to consider the energy of the battery itself

$$dW = F\,dx + V\,dQ$$

$$\Rightarrow \quad F = -\frac{dW}{dx} + V\frac{dQ}{dx}$$

$$\Rightarrow \quad F = -\frac{dW}{dx} + V^2\frac{dc}{dx}$$

$$= -\left(\frac{1}{2}V^2\frac{dc}{dx}\right) + V^2\frac{dc}{dx}$$

$$= \frac{1}{2}V^2\frac{dC}{dx} = \frac{1}{2}V^2\frac{w(\epsilon-1)}{4\pi d}$$

$$= \frac{V^2 w(\epsilon-1)}{8\pi d}$$ ✓

Notice: the force is const (independent of x)

or, Find $W = \frac{1}{4\pi}\int \bar{E}\cdot\bar{D}\,d^3x$

and $F = -\frac{dW}{dx}$

c) $Q = CV$

$$\Rightarrow \frac{dQ}{dt} = \frac{d}{dt}(CV) = \frac{dx}{dt}\frac{d}{dx}(CV) = v_0 V \frac{dC}{dx}$$

$$\Rightarrow I = v_0 \frac{V\omega(\epsilon-1)}{4\pi d}$$

d) Const acceleration. Velocity increases linearly

the potential energy stored in the slab

$$U = F \cdot X$$

$$U = \frac{V^2\omega(\epsilon-1)x}{8\pi d}$$

↑ length of the slab inside the capacitor

when K.E is max, all potential energy is converted to kinetic energy

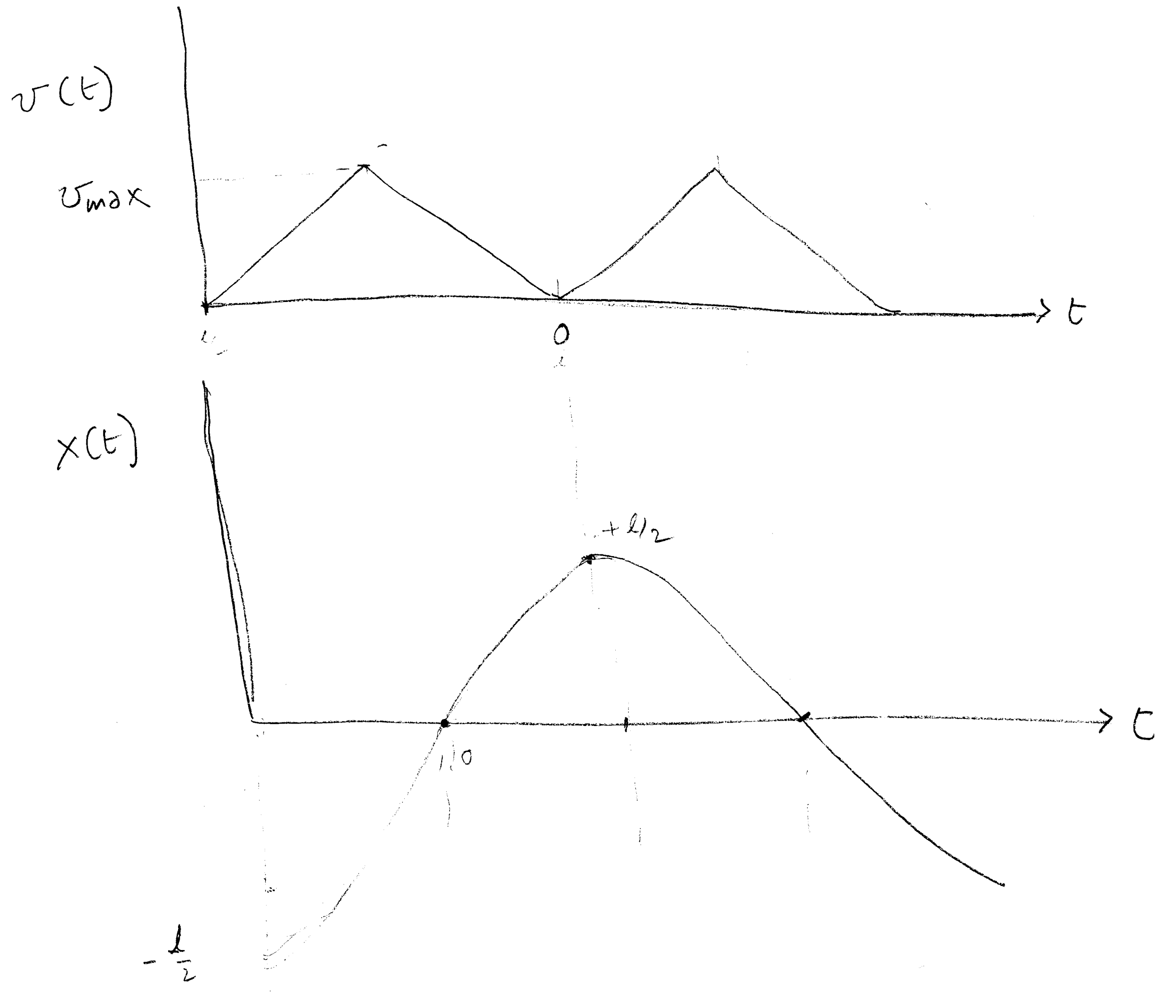
$$\frac{1}{2}mv^2 = \frac{V^2\omega(\epsilon-1)x}{8\pi d}$$

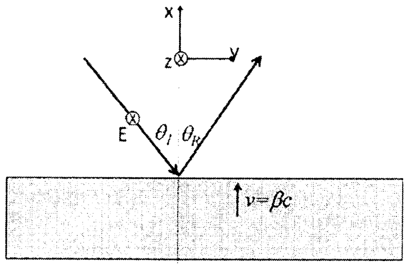
but $x_{max} = l$

$$\Rightarrow \frac{1}{2}m v_{max}^2 = \frac{V^2\omega(\epsilon-1)l}{8\pi d}$$

$$\Rightarrow v_{max} = V\left(\frac{\omega(\epsilon-1)l}{m\,4\pi d}\right)^{1/2}$$

e)





2. This problem investigates the shifting frequency of electromagnetic radiation that is reflected off a moving target. Incident and reflected frequencies and angles are not the same if the target is moving.

   Assume that in the lab frame of reference, the target is a flat mirror traveling upward in the positive x-direction parallel to the mirror's normal with velocity $\mathbf{v} = \beta c\,\hat{\mathbf{x}}$ (see the figure). Also assume the wave is a linearly polarized plane wave traveling in vacuum towards the moving mirror at angle θ_I (relative to the mirror's normal). If the polarization is in the $\hat{z}$ direction, the incident electric field is given by

$$\mathbf{E}_I = E_0\,\hat{\mathbf{z}}\,e^{i(\mathbf{k}_I\cdot\mathbf{r}-\omega_I t)},$$

   with

$$\mathbf{k}_I = \frac{\omega_I}{c}(-\cos\theta_I\,\hat{\mathbf{x}} + \sin\theta_I\,\hat{\mathbf{y}}).$$

   (a) {2 pts} Write the Lorentz boost A as a function of β and $\gamma \equiv \sqrt{1-\beta^2}$ that transforms the Lab coordinates $\mathbf{r}$ and ct to coordinates $\mathbf{r}'$ and ct' co-moving with the mirror. Also give the inverse A^{-1} of the Lorentz boost A that transforms the moving coordinates $\mathbf{r}'$ and ct' into Lab coordinates $\mathbf{r}$ and ct.

   (b) {3 pts} By rewriting the above wave's phase in both reference frames, i.e.,

$$\mathbf{k}_I\cdot\mathbf{r} - \omega_I t = \mathbf{k}'_I\cdot\mathbf{r}' - \omega'_I t'$$

   as a function of the co-moving mirror coordinates $\mathbf{r}'$ and ct' (i.e., use A^{-1}) find $\mathbf{k}'_I$ and ω'_I as observed in the co-moving frame. These will be functions of β, γ, and θ_I as well as ω_I.

   (c) {2 pts} By writing the incident wave vector just obtained in the moving frame in the form

$$\mathbf{k}'_I = \frac{\omega'_I}{c}(-\cos\theta'_I\,\hat{\mathbf{x}} + \sin\theta'_I\,\hat{\mathbf{y}}),$$

   determine the incident angle θ'_I as seen by observers moving with the mirror (e.g., give $\cos\theta'_I$ as a function of θ_I, ω_I and the Lorentz parameters β, γ).

(d) {3 pts} If, as seen by observers moving with the mirror, the reflected wave has the same frequency as the incident wave $\omega'_R = \omega'_I$ and a reflection angle that is the same as the incidence angle $\theta'_R = \theta'_I$, i.e.,

$$\mathbf{k}'_R = \frac{\omega'_I}{c}(\cos\theta'_I\,\hat{\mathbf{x}} + \sin\theta'_I\,\hat{\mathbf{y}}),$$

what is the frequency ω_R of the reflected light as measured in the laboratory frame? Hint: again use

$$\mathbf{k}_R \cdot \mathbf{r} - \omega_R t = \mathbf{k}'_R \cdot \mathbf{r}' - \omega'_R t',$$

and the Lorentz boosts A.

Prob 2 (Gaussian)

a)
$$A = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

b)
$$\bar{E}_I = \hat{z}\, E_0\, e^{i(\bar{K}_I \cdot \bar{r} - \omega_I t)}$$

$$\bar{K}_I = \frac{\omega_I}{c}\left(-\cos\theta_I\, \hat{x} + \sin\theta_I\, \hat{y}\right)$$

$$\bar{E}' = E_{\parallel} + \gamma\left(E_{\perp} + \bar{\beta}\wedge\bar{B}\right)$$

$$\bar{E}'(r) = \gamma E_0\, e^{i(\bar{K}_I\cdot\bar{r} - \omega_I t)}$$

$$\rightsquigarrow \quad \bar{K}_I\cdot\bar{r} - \omega_I t = \bar{K}'_I\cdot\bar{r}' - \omega'_I t'$$

$$\Rightarrow \bar{K}_I\cdot\left(\bar{r}'_{\perp} + \gamma(\bar{\beta}ct' + \bar{r}'_{\parallel})\right) - \omega_I\left(\gamma c t' + \gamma\bar{\beta}\,\bar{r}'_{\parallel}\right)$$

$$= \bar{K}'_I\cdot\bar{r}' - \omega_I t'$$

$$\Rightarrow \quad \bar{K}_I \cdot \bar{r}_\perp' + \gamma \bar{K}_I \cdot (\bar{\beta} c t') + \gamma (\bar{K}_I \cdot \bar{r}_\parallel')$$

$$- \omega_I \gamma c t' - \gamma \omega_I \bar{\beta} \cdot \bar{r}_\parallel' = \bar{K}_I' \cdot \bar{r}' - \omega_I' t'$$

$$\Rightarrow \quad \bar{K}_I \cdot \bar{r}_\perp' + (\gamma \bar{K}_I - \gamma \omega_I \bar{\beta}) \cdot \bar{r}_\parallel' = \bar{K}_I' \cdot \bar{r}_\perp' + \bar{K}_I' \cdot \bar{r}_\parallel' - \omega_I' t'$$

$$+ (\gamma \bar{K}_I \cdot \bar{\beta} c - \omega_I \gamma c) t'$$

$$\Rightarrow \quad \bar{K}_I' = \gamma (\bar{K}_I - \omega_I \bar{\beta})$$

$$\omega_I' = \gamma (\bar{K}_I \cdot \bar{\beta} c - \omega_I c)$$

c) $$\bar{K}_I' = \gamma \left\{ |K_I| (-\cos\theta_I' \hat{x} + \sin\theta_I' \hat{y}) - \omega_I \beta \hat{x} \right\}$$

$$= \frac{\omega_I}{c} \left\{ \gamma (-\cos\theta_I' \hat{x} + \sin\theta_I' \hat{y}) - \gamma \beta c \hat{x} \right\}$$

$$= \frac{\omega_I}{c} \left\{ (-\gamma \cos\theta_I' - \gamma \beta c) \hat{x} + \gamma \sin\theta_I' \hat{y} \right\}$$

b) $$\bar{K}_I = |K_I|(-\cos\theta_I \hat{x} + \sin\theta_I \hat{y})$$

$$\bar{K}_I \cdot \bar{r} = \frac{\omega_I}{c}(-x\cos\theta_I + y\sin\theta_I)$$

$$\Rightarrow \quad \bar{K}_I \cdot \bar{r} - \omega_I t = \bar{K}_I' \cdot \bar{r}' - \omega_I' t'$$

$$\Rightarrow -\frac{\omega_I}{c} x\cos\theta_I + \frac{\omega_I}{c} y \sin\theta_I - \omega_I t$$
$$= (K_I')_x x' + (K_I')_y y' + (K_I')_z z' - \omega_I' t'$$

$$\Rightarrow -\frac{\omega_I}{c}\left(\gamma(\beta c t' + x')\right)\cos\theta_I + \frac{\omega_I}{c} y' \sin\theta_I$$
$$- \frac{\omega_I}{c}\left(\gamma(ct' + \beta x')\right) = (\quad)$$

$$\Rightarrow \left(-\frac{\omega_I}{c}\gamma(\cos\theta_I + \beta)\right)x' + \left(\frac{\omega_I}{c}\sin\theta_I\right)y'$$
$$- \frac{\omega_I \gamma \not{c}}{\not{c}}(\beta\cos\theta_I + 1)t' = (\ )$$

$$(K_I')_x = -\frac{\omega_I}{c}\gamma(\cos\theta_I + \beta)$$

$$(K_I')_y = \frac{\omega_I}{c}\sin\theta_I$$

$$(K_I')_z = 0$$

$$\omega_I' = \gamma\omega_I(1 + \beta\cos\theta_I)$$

$$\omega_I = \frac{\omega_I'}{\gamma(1 + \beta\cos\theta_I)}$$

$$\vec{K}_I' = -\frac{\omega_I'}{c(1+\beta\cos\theta_I)}(\cos\theta_I+\beta)\,\hat{x} + \frac{\omega_I'}{\gamma c(1+\beta\cos\theta_I)}\,\hat{y}\,\sin\theta_I$$

$$= \frac{\omega_I'}{c}\left(-\frac{(\cos\theta_I+\beta)}{1+\beta\cos\theta_I}\,\hat{x} + \frac{\sin\theta_I}{\gamma(1+\beta\cos\theta_I)}\,\hat{y}\right)$$

$$\cos\theta_I' = \frac{\cos\theta_I+\beta}{1+\beta\cos\theta_I}$$

$$\sin\theta_I' = \frac{\sin\theta_I}{\gamma\ 1+\beta\cos\theta_I}$$

check: $\cos^2\theta_I' + \sin^2\theta_I \stackrel{!}{=} 1$

$$\Rightarrow \frac{\cos^2\theta_I + \beta^2 + 2\beta\cos\theta_I + \gamma^{-2}\sin^2\theta_I}{(1+\beta\cos\theta_I)^2} \qquad \beta^2 = 1-\gamma^{-2}$$

$$= \frac{\cos^2\theta_I + \sin^2\theta_I - \beta^2\sin^2\theta_I + \beta^2 + 2\beta\cos\theta_I}{(\ )^2}$$

$$= \frac{1+\beta^2\cos^2\theta + 2\beta\cos\theta_I}{(\ )^2} = 1\ \checkmark$$

d)

$$\bar{K}_R' = \frac{\omega_I'}{c}\left(\cos\theta_I' \hat{x} + \sin\theta_I' \hat{y}\right)$$

$$\vec{k}_R' \cdot \bar{r}' - \omega_R' t' = \bar{K}_R \cdot \bar{r} - \omega_R t$$

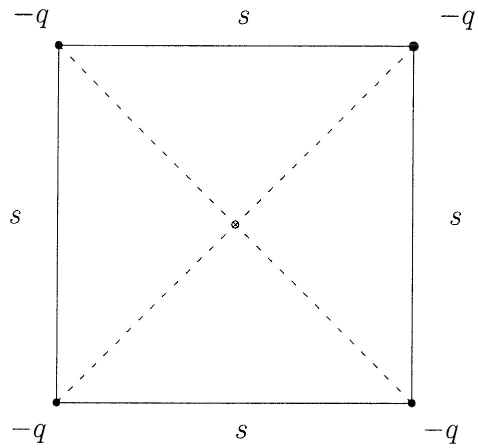
$$\Rightarrow \frac{\omega_I'}{c}\left(x'\cos\theta_I' + y'\sin\theta_I' - ct'\right) = (K_R)_x x + (K_R)_y Y - \omega_R t$$

$$\Rightarrow \frac{\omega_I'}{c}\left\{ (\gamma(-\beta ct + x)\cos\theta_I' + Y\sin\theta - \gamma(ct - \beta x) \right\} = (\quad)$$

$$\Rightarrow \frac{\omega_I'}{c}\left\{ \gamma(\cos\theta_I' + \beta)x + y\sin\theta + \gamma(-\beta c\cos\theta_I' - c) \right\} = (\quad) - \omega_R t$$

$$\Rightarrow \omega_R = \frac{\gamma\omega_I' c}{c}\left(\beta\cos\theta_I' + 1\right)$$

$$= \gamma^2 \omega_I (1 + \beta\cos\theta_I)\left(1 + \frac{\beta(\cos\theta_I + \beta)}{1 + \beta\cos\theta_I}\right)$$



3. Consider a square with sides of length s and charges -q at the corners as shown:

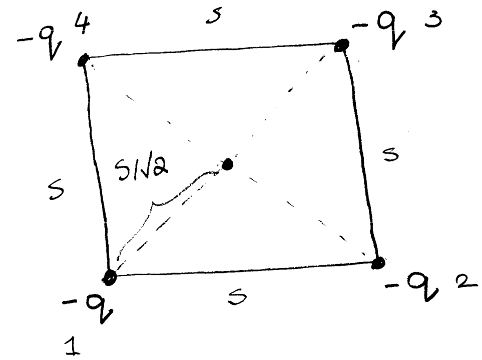
   (a) {2 pts} What is the potential at the center of the square if the potential is zero at ∞?

   (b) {2 pts} How much work does it take to bring in another charge -q from ∞ to the center of the square?

   (c) {3 pts} How much work does it take to assemble the original configuration of 4 negative charges (no charge at center)?

   (d) {3 pts} Now suppose that instead of the 4 charges being located at the corners of a square, a net charge of $-4q$ is distributed uniformly on the surface of a sphere of radius s. How much work does it take to bring in another charge q from ∞ to the center of the sphere?

Prob 3 (Gaussian)



a) $\Phi_1(r) = \frac{-q}{|r-\bar{r}'|} = -\frac{\sqrt{2}\,q}{s}$

$\Phi_{net}(0) = -\frac{4\sqrt{2}\,q}{s}$

b) $W = q\,\Phi_{net}(0)$

$W = +\frac{4\sqrt{2}\,q^2}{s}$

c) $W_1 = 0$

$W_2 = -\frac{q}{s}(-q) = \frac{q^2}{s}$

$W_3 = \frac{q^2}{\sqrt{2}s} + \frac{q^2}{s} = \frac{q^2}{s}\left(1 + \frac{1}{\sqrt{2}}\right)$

$W_4 = \frac{q^2}{s} + \frac{q^2}{s} + \frac{q^2}{\sqrt{2}s} = \frac{q^2}{s}\left(2 + \frac{1}{\sqrt{2}}\right)$

$W_{tot} = \frac{q^2}{s}\left(4 + \sqrt{2}\right)$

d) $\sigma = \frac{-4q}{4\pi s^2}$

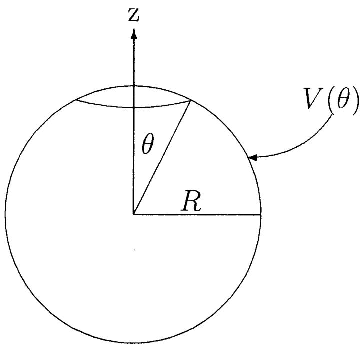
$E(4\pi r^2) = 4\pi Q_{enc}$

$\bar{E} = \frac{-4q}{r^2}\hat{r} \quad r > s \qquad \bar{E} = 0 \quad r < s$

$\bar{F}_q = q\bar{E} = -\frac{4q^2}{r^2}\hat{r}$

$W = \int_{\infty}^{0} \bar{F}.d\bar{r} = \int_{\infty}^{s} -\frac{4q^2}{r^2}dr = \left[\frac{4q^2}{r}\right]_{\infty}^{s}$

$W = \frac{4q^2}{s}$



4. Consider an isolated spherical surface of radius R centered on the origin, that is kept at a known potential $V(\theta)$, i.e.,

$$\Phi(r = R, \theta) = V(\theta)$$

where (r, θ, ϕ) are the usual spherical polar coordinates, i.e., θ is measured with respect to a z-axis passing through the center of the sphere and ϕ is the azimuthal angle about the z-axis measured from the x axis.

(a) {2 pts} Write down expressions for the general solution to $\nabla^2\Phi(r, \theta) = 0$ for the electrostatic potential as a linear combination of Legendre polynomials in the respective regions $0 \leq r < R$ and $r > R$. Assume that the potential vanishes at $r \to \infty$ and has azimuthal symmetry i.e., no dependence on the angle ϕ. Do not include terms that must vanish. Do not attempt to evaluate the constants that appear in the linear combination but do give the correct r dependence of each term.

(b) {2 pts} What boundary conditions must your two expressions satisfy at the junction $r = R$ to have a unique solution to Maxwell's equations?

(c) {2 pts} If the particular surface potential imposed is

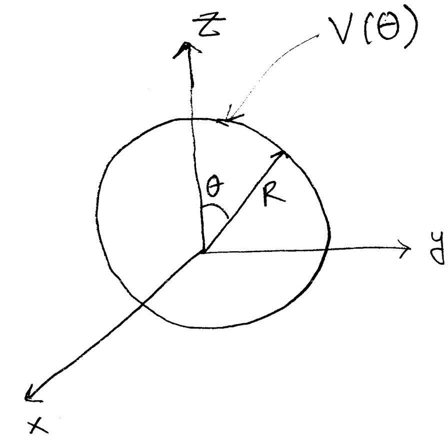
$$\Phi(r = R, \theta) = V_0 \cos\theta$$

where V_0 is a constant, what is the explicit form of your potential for both regions $r \leq R$ and $r > R$?

(d) {2 pts} Determine the resulting electric field on both sides of the $r{=}R$ surface.

(e) {2 pts} What is the surface charge density $\sigma(\theta)$ on the spherical shell at $r{=}R$.

Prob 4 (Gaussian)



a) The General Solution of

$$\nabla^2 \Phi(r,\theta) = 0$$

in spherical co-ordinate

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)$$

where, for $\underline{0 \leq r < R}$

$$\Phi(r,\theta)\Big|_{in} = \sum_{l=0}^{\infty} A_l r^l \, P_l(\cos\theta)$$

and for $\underline{r > R}$

$$\Phi(r,\theta)\Big|_{out} = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

b) i) $\Phi_{in}(R,\theta) = \Phi_{out}(r=R,\theta) = V(\theta)$

and, ii) $\left.\dfrac{\partial \Phi_{in}}{\partial r}\right|_{r=R} - \left.\dfrac{\partial \Phi_{out}}{\partial r}\right|_{r=R} = 4\pi\sigma$

c)

i) $$\sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = \sum_{l=0}^{\infty} B_l R^{-(l+1)} P_l(\cos\theta)$$

$$\Rightarrow \quad A_l = B_l R^{-(2l+1)}$$

/* ii)

$$\sum_{l=0}^{\infty} l A_l R^{l-1} P_l(\cos\theta) = -\sum_{l=0}^{\infty} B_l(l+1) R^{-(l+2)} P_l(\cos\theta)$$

$$\Rightarrow \sum_{l=0}^{\infty} l B_l R^{-(l+2)} P_l(\cos\theta) = \sum_{l=0}^{\infty} (l+1) B_l R^{-(l+2)} P_l(\cos\theta)$$

$$\Rightarrow \sum$$ */

$$\rightsquigarrow \sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = V_0 \cos\theta = V_0 P_1(\cos\theta)$$

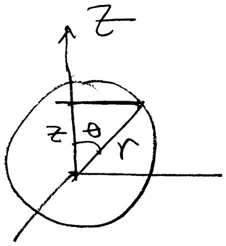
Using Orthogonality of Legendre Polynomials

$$\sum_{l=0}^{\infty} A_l R^l \int P_l(\cos\theta) P_{l'}(\cos\theta)\, d(\cos\theta)$$

$\rightsquigarrow l = 1,$ $\Rightarrow A_1 R = V_0$

$$\Rightarrow \quad A_1 = \frac{V_0}{R}$$

$$B_1 = A_1 R^3 = V_0 R^2$$



$$\Phi\big|_{r>R}(r,\theta) = \frac{V_0}{r^2} R^2 \cos\theta = \frac{V_0 R^2 z}{r^3}$$

$$\Phi\big|_{r<R}(r,\theta) = \frac{V_0}{R} r \cos\theta = \frac{V_0 z}{R}$$

d) $$E\big|_{r<R} = -\nabla\Phi = \frac{\partial}{\partial r}\left(\frac{V_0}{R} r\cos\theta\right) + \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{V_0}{R} r\cos\theta\right)$$

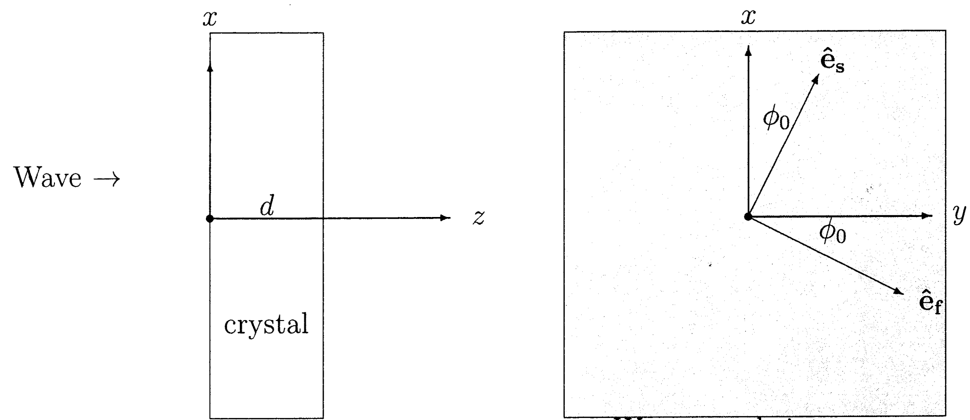
$$= \hat{r}\frac{V_0}{R}\cos\theta - \hat{\theta}\frac{V_0}{R}\sin\theta = \frac{V_0}{R}(\cos\theta - \sin\theta)$$

$$E\big|_{r>R} = -\nabla\Phi = \hat{r}\frac{\partial}{\partial r}\left(\frac{V_0}{r^2} R^2\cos\theta\right) + \frac{\hat{\theta}}{r}\frac{\partial}{\partial\theta}\left(\frac{V_0}{r^2} R^2\cos\theta\right)$$

$$= \hat{r}\left(-\frac{2V_0}{r^3} R^2\cos\theta\right) + \hat{\theta}\left(-\frac{V_0}{r^3} R^2\sin\theta\right)$$

d) $$\hat{r}\left(-\frac{2V_0}{r^3} R^2\cos\theta\right) + \hat{\theta}\left(-\frac{V_0}{r^3} R^2\sin\theta\right) - \left(\frac{V_0}{R}\cos\theta\right)\hat{r}$$

$$+ \hat{\theta}\left(\frac{V_0}{R}\sin\theta\right) = 4\pi\sigma$$



Wave travels into page.

5. A plane polarized monochromatic light wave traveling in the $+z$ direction enters a large flat slab of transparent crystal of thickness d, located between $z = 0$ and $z = d$. This crystal has the property that the index of refraction depends on the direction of polarization as follows: Plane waves traveling in the z direction but polarized in the direction

$$\hat{\mathbf{e}}_s = \cos\phi_0\hat{\mathbf{x}} + \sin\phi_0\hat{\mathbf{y}},$$

travel with speed $v_s = c/n_s < c$ but those polarized in the orthogonal direction

$$\hat{\mathbf{e}}_f = -\sin\phi_0\hat{\mathbf{x}} + \cos\phi_0\hat{\mathbf{y}},$$

travel with the faster speed $v_f = c/n_f < c$ where $n_s = n_f + \delta n$.

Assume the wave, just after entering the crystal (i.e., for very small $z << \lambda < d$), is polarized in the y direction and hence has the form

$$\mathbf{E}(z \approx 0, t) = E_0\,\hat{\mathbf{y}}\,e^{-i\omega t}.$$

(a) {4 pts} Prove that in general the initial plane wave becomes elliptically polarized when it reaches $z = d$ by deriving the following expression

$$\mathbf{E}(z = d, t) = [E_x\,\hat{\mathbf{x}} + E_y\,\hat{\mathbf{y}}]\,e^{i(\bar{k}d - \omega t)},$$

where

$$\bar{k} \equiv \frac{\omega}{c}\left(\frac{n_s + n_f}{2}\right),$$

and

$$E_x = iE_0 \sin 2\phi_0 \sin\delta,$$
$$E_y = E_0(\cos\delta - i\cos 2\phi_0 \sin\delta),$$

with

$$\delta \equiv \frac{\omega d}{2c}\delta n.$$

Hint: Write the wave at z=0 as a combination of slow and fast plane polarized parts using $\hat{\mathbf{y}} = \sin\phi_0\hat{\mathbf{e}}_s + \cos\phi_0\hat{\mathbf{e}}_f$.

(b) {3 pts} For what values of δ and θ_0 will the wave emerge from the crystal as a circularly polarized wave? $(E_x/E_y = \pm i)$.

(c) {3 pts} For what minimum crystal thicknesses $d = d_{min}$ will the wave emerge as a plane polarized wave $(E_x/E_y = \text{real})$ and what will its polarization direction be?

Prob 5 (Gaussian)

$$E(z\approx 0,t) = E_0\,\hat{y}\,e^{-i\omega t}$$
$$= E_0(\sin\phi_0\,\hat{e}_s + \cos\phi_0\,\hat{e}_f)\,e^{-i\omega t}$$
$$= E_0\sin\phi_0\,e^{-i\omega t}\,\hat{e}_s + E_0\cos\phi_0\,e^{-i\omega t}$$

$$E(z=d,t) = \underbrace{E_0\sin\phi_0\,e^{i(k_s d}}\,{}^{-\omega t)}\,\hat{e}_s + \underbrace{E_0\cos\phi_0\,e^{i(k_f d}}\,{}^{-\omega t}\,\hat{e}_f$$

$$= \sqcup\,(\cos\phi_0\,\hat{x} + \sin\phi_0\,\hat{y}) + \sim(-\sin\phi_0\,\hat{x} + \cos\phi_0\,\hat{y})$$

$$= E_0\sin\phi_0\cos\phi_0\left\{e^{ik_s d - i\omega t} - e^{ik_f d - i\omega t}\right\}\hat{x}$$
$$+ E_0\,e^{-i\omega t}\left\{\sin^2\phi_0\,e^{ik_s d} + \cos^2\phi_0\,e^{ik_f d}\right\}\hat{y}$$

$$\text{1st term} = \underbrace{E_0\sin\phi_0\cos\phi_0\,e^{-i\omega t}}\left\{e^{i\frac{\omega}{c}n_s d} - e^{i\frac{\omega}{c}n_f d}\right\}$$
$$= \sim\; e^{i\frac{\omega}{c}\left(\frac{n_s+n_f}{2}\right)d}\left\{e^{i\frac{\omega}{c}\left(\frac{n_s-n_f}{2}\right)d} - e^{i\frac{\omega}{c}\left(\frac{n_f-n_s}{2}\right)d}\right\}$$
$$= E_0\sin\phi_0\cos\phi_0\,e^{i(\bar{k}\cdot d-\omega t)}\left\{e^{i\frac{\omega\delta u}{c}} - e^{-i\frac{\omega\delta u}{c}}\right\}$$
$$E_x = i E_0\sin 2\phi_0\,\sin\delta\,\hat{x}$$

2nd term = $E_0 e^{-i\omega t}\left\{\sin^2\phi_0\, e^{ik_s d} + \cos^2\phi_0\, e^{ik_f d}\right\}$

$$= E_0\, e^{i\left(\frac{k_s+k_f}{2}\right)d - i\omega t}\left\{\sin^2\phi_0\, e^{i\left(\frac{k_s-k_f}{2}\right)d} + \cos^2\phi_0\, e^{i\left(\frac{k_f-k_s}{2}\right)d}\right\}$$

$$= E_0 e^{i(\bar{k}d-\omega t)}\left\{\sin^2\phi_0\, e^{i\frac{\omega\delta n}{2c}} + \cos^2\phi_0\, e^{-i\frac{\omega\delta n}{2c}}\right\}$$

$$= E_0 e^{i(\bar{k}d-\omega t)}\left\{\tfrac{1}{2}(1-\cos 2\phi_0)\, e^{i\delta} + \tfrac{1}{2}(1+\cos 2\phi_0)\, e^{-i\delta}\right\}$$

$$E_y = E_0 e^{i(\bar{k}d-\omega t)}\left\{\cos\delta - i\cos 2\phi_0 \sin\delta\right\}\hat{y}$$

$$\therefore\ \bar{E}(z=d,t) = \left[E_x\hat{x} + E_y\hat{y}\right]e^{i(\bar{k}d-\omega t)}$$

b) $$\frac{E_x}{E_y} = \frac{i E_0 \sin 2\phi_0 \sin\delta}{E_0(\cos\delta - i\cos 2\phi_0 \sin\delta)} = i\,\frac{\sin 2\phi_0}{\cot\delta - i\cos 2\phi_0}$$

$\delta = \pi/4,\ \phi_0 = \pi/4$

$$\frac{E_x}{E_y} = i\,\frac{1}{1} = i$$

$\delta = \pi/4,\ \phi_0 = 3\pi/4$

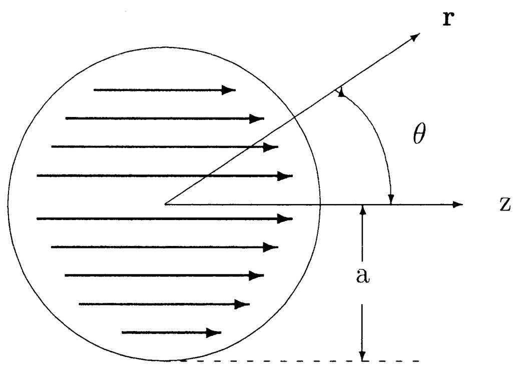
$$\frac{E_x}{E_y} = -i$$

c) $\delta = \pi/2$

$$\frac{E_x}{E_y} = -\tan\phi_0 \Rightarrow \text{real}$$

$$\pi/2 = \frac{\omega d}{2c}\delta_n = \frac{k_s - k_f}{2} d$$

$$\Rightarrow \quad d = \frac{\pi}{k_s - k_f}$$



6. A permanent magnet in the shape of a solid sphere of radius a is oriented on the z-axis as shown in the figure. The magnetization of the magnet is given by $\vec{M} = M_0\hat{z}$. [Recall that $\nabla \times \mathbf{H} = 0$ implies the existance of a magnetic scalar potential $\Phi_m(r,\theta)$ related to the magnetic field by $\mathbf{H} = -\vec{\nabla}\Phi_m(r,\theta)$.]

   (a) {4 pts} Compute the scalar magnetic potential $\Phi_m(r,\theta)$ at all points $r < a$ and $r > a$.

   (b) {3 pts} Compute the magnetic Field $\mathbf{H} = -\vec{\nabla}\Phi_m(r,\theta)$ at all points $r < a$ and $r > a$.

   (c) {3 pts} Compute the magnetic induction $\mathbf{B}$, where

$$\begin{aligned} \mathbf{B}/\mu_0 &= \mathbf{H} + \mathbf{M}, & (SI) \\ \mathbf{B} &= \mathbf{H} + 4\pi\mathbf{M}, & (Gaussian) \end{aligned}$$

   at all points $r < a$ and $r > a$.

Hints: The magnetic potential is axial symmetric about the z-axis and satisfies the Laplace equation at all points except $r = a$. Legendre polynomials are useful.