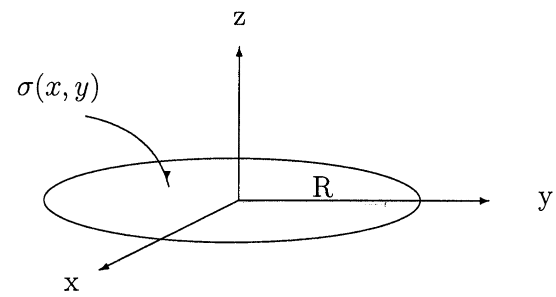
E & M Qualifier

January 14, 2010

To insure that the your work is graded correctly you MUST:

1. use only the blank answer paper provided,
2. write only on one side of the page,
3. put your alias on every page,
4. put the problem # on every page,
5. start each problem by stating your units e.g., SI or Gaussian,
6. number every page starting with 1 for each problem,
7. put the total # of pages you use for that problem on every page,
8. staple your exam when done.

Use only the reference material supplied (Schaum's Guides).



1. Consider a thin nonconducting disk of radius R centered on the origin of a coordinate system, lying in the x-y plane, and carrying a surface charge density given by

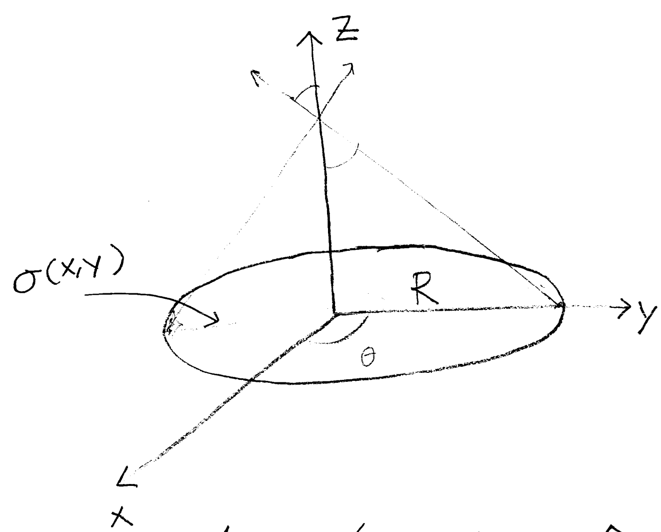
$$\sigma = \sigma_o \frac{yR}{x^2 + y^2}.$$

   (a) {6 pts} Determine the electric field at a location $\vec{r} = z\hat{k}$.

   (b) {3 pts} Give an approximation to your answer to part (a) that is valid for the $z >> R$.

   (c) {1 pts} Find the force on a charge q located at a position $\vec{r} = z\hat{k}$.

Prob 1



a) $$\bar{E} = \int \frac{\sigma \, d^2x'}{|\bar{r}-\bar{r}'|^2}$$

$$\bar{r} = z\,\hat{z}$$

$$\bar{r}' = x'\hat{x} + y'\hat{y}$$

$$\bar{r} - \bar{r}' = -x'\hat{x} - y'\hat{y} + z\hat{z}$$

$$= \int \frac{\sigma\,(-x'\hat{x} - y'\hat{y} + z\hat{z})}{(x'^2+y'^2+z^2)^{3/2}}\, dx'dy'$$

$$= -\int \frac{\sigma_0\, y'x'R \; dx'dy'}{(x'^2+y'^2)(x'^2+y'^2+z^2)^{3/2}}\,\hat{x} - \int \frac{\sigma_0 R y'^2 \, dx'dy'}{(x'^2+y'^2)(x'^2+y'^2+z^2)^{3/2}}\,\hat{y}$$

$$+ \int \frac{\sigma_0 R\, z y' \, dx'dy'}{(x'^2+y'^2)(x'^2+y'^2+z^2)^{3/2}}\,\hat{z}$$

$$= -\int \frac{\sigma_0 R\, r'^2 \sin\theta'\cos\theta'}{r'^2(r'^2+z^2)^{3/2}}\, r'dr'd\theta'\,\hat{x} - \int \frac{\sigma_0 R r'^2 \sin^2\theta' \, r'dr'd\theta'}{r'^2(r'^2+z^2)^{3/2}}$$

$$+ \int \frac{\sigma_0 R z r' \sin\theta'}{r'^2(r'^2+z^2)^{3/2}}\, r'dr'd\theta'\,\hat{z}$$

but $\int_0^{2\pi} \sin\theta' \cos\theta' d\theta' = \int_0^{2\pi} \sin\theta' d\theta' = 0$

$$\bar{E} = -\pi\sigma_0 R \int_0^R \frac{r' dr'}{(r'^2+z^2)^{3/2}} \hat{y}$$

$$= -\pi\sigma_0 R \left[-\frac{1}{\sqrt{r'^2+z^2}} \right]_0^R \hat{y}$$

$$\bar{E}(z) = \pi\sigma_0 \left[\frac{R}{\sqrt{R^2+z^2}} - \frac{R}{z} \right] \hat{y}$$

b) for $z >> R$

$$\left(\frac{R^2}{z^2}+1\right)^{-1/2} = \left(\frac{R^2}{z^2}+1\right)^{-\frac{1}{2}}$$

$$\approx \left(1 - \frac{R^2}{2z^2}\right)$$

$$\bar{E}(z) = \pi\sigma_0 \left[\frac{R}{z}\left(1-\frac{R^2}{2z^2}\right) - \frac{R}{z} \right] \hat{y}$$

$$= -\frac{\pi\sigma_0 R^3}{2z^3}$$

c) $\bar{F} = q\bar{E}(z) = q\pi\sigma_0 R \left[\frac{1}{\sqrt{R^2+z^2}} - \frac{1}{z} \right] \hat{y}$

S-10

?

2. Consider a linear, homogeneous, isotropic, and non-dissipative dielectric (i.e., a dielectric where $\mathbf{D} = \epsilon\mathbf{E}$ and ϵ is a constant) in the shape of a sphere of radius R with a point charge Q embedded at its center.

   (a) {2 pts} Find the electric displacement vector $\mathbf{D}$, the electric field $\mathbf{E}$, and the polarization density $\mathbf{P}$ inside the dielectric.

   (b) {2 pts} Find the bound charge volume density ρ_D inside the dielectric.

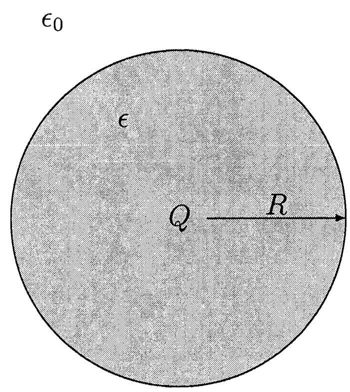
   (c) {1 pts} Find the total bound charge Q_D on the $r = R$ boundary of the dielectric.

   (d) {2 pts} Find the net charge (free plus bound) at the center of the dielectric.

   (e) {1 pts} Find the electric displacement vector $\mathbf{D}$, the electric field $\mathbf{E}$, and the polarization density $\mathbf{P}$, outside the dielectric sphere.

   (f) {2 pts} Are $\mathbf{D}$ and $\mathbf{E}$ continuous at $r = R$? If not explain why.

   (If you use Gaussian units you can put $\epsilon_0 = 1$.)



Prob 2 (Gaussian)

a) $\int \bar{D} \cdot d\bar{a} = 4\pi Q_{fenc}$

$\Rightarrow D \, 4\pi r^2 = 4\pi Q$

$\Rightarrow \bar{D} = \frac{Q}{r^2} \hat{r}$

$\bar{E} = \frac{Q}{\epsilon r^2} \hat{r}$

$\bar{P} = \frac{1}{4\pi} (\bar{D} - \bar{E}) = \frac{Q}{4\pi r^2} \left(1 - \frac{1}{\epsilon}\right) \hat{r}$

b) $\rho_b = -\bar{\nabla} \cdot \bar{P} = -\frac{1}{r^2 \sin\theta} \frac{\partial}{\partial r} \left(r^2 \sin\theta \frac{1}{r^2}\right) \frac{1 - \frac{1}{\epsilon}}{4\pi}$

$\Rightarrow \rho_b = 0$

c) $\sigma_b = \bar{P} \cdot \hat{r} \Big|_{r=R} = \frac{Q}{4\pi R^2} \left(1 - \frac{1}{\epsilon}\right)$

$Q_b = \int \sigma_b \, d^2x = \int \frac{Q}{4\pi R^2} \left(1 - \frac{1}{\epsilon}\right) R^2 \sin\theta \, d\theta \, d\phi$

$= Q \left(1 - \frac{1}{\epsilon}\right)$

e) $\int \bar{D} \cdot d\bar{a} = 4\pi Q_{f\,enc}$

$D\,4\pi r^2 = 4\pi Q$

$\Rightarrow \bar{D} = \frac{Q}{r^2}\hat{r}$

$\bar{E} = \frac{Q}{r^2}\hat{r}$

$\bar{P} = 0$

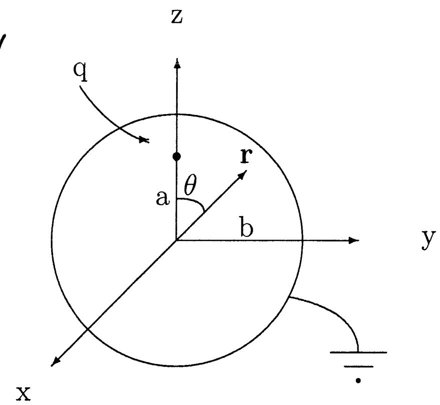
f) $D^{\perp}_{above} - D^{\perp}_{below} = 4\pi \sigma_f = 0$

so $\bar{D}$ continous

$E^{\perp}_{above} - E^{\perp}_{below} = 4\pi \sigma_{tot} = 4\pi\sigma_b$

$\Rightarrow \bar{E}$ not continuous

√



3. A thin grounded hollow conducting sphere of radius 'b' is centered at the origin. A point charge q is located on the z-axis at $z = a < b$ INSIDE the sphere.

   (a) {5 pts} Write the total potential for this system as a sum,

$$\Phi = \Phi_{sphere} + \Phi_q,$$

   where Φ_q is the potential due to the point charge and Φ_{sphere} (in spherical polar coordinates) is the appropriate linear combination of Legendre polynomials $P_\ell(\cos(\theta))$. Evaluate the coefficients of the $P_\ell(\cos(\theta))$ in the Φ_{sphere} expansion. Recall that the Legendre polynomials are independent orthogonal functions satisfying

$$\int_{-1}^{1} P_\ell(x) P_{\ell'}(x)\, dx = \frac{2}{2\ell + 1}\, \delta_{\ell\ell'}$$

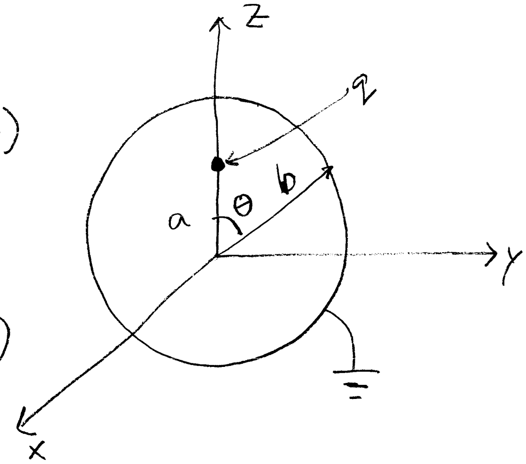
   and

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \sum_{\ell=0}^{\ell=\infty} \frac{(r_<)^\ell}{(r_>)^{\ell+1}} P_\ell(\cos(\gamma))$$

   where γ is the angle between the two directions $\mathbf{r}$ and $\mathbf{r}'$.

   (b) {5 pts} Show that your expression for Φ_{sphere} is equivalent to the potential of a point charge. Where is the point charge located and what is it's charge?

Prob 3 (Gaussian)



(a) $$\Phi_{sphere}(r,\theta) = \sum_{l=0}^{\infty}\left[A_l r^l + \frac{B^l}{r^{l+1}}\right] P_l(\cos\theta)$$

$$\Phi_q(r,\theta) = \frac{q}{|\bar{r}-\bar{r}'|} = q\sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\theta)$$

• Inside,

$$\Phi_{sphere}(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

$$\Phi_q(r,\theta)\Big|_{a<r<R} = q\sum_{l=0}^{\infty} \frac{a^l}{r^{l+1}} P_l(\cos\theta)$$

$$\rightsquigarrow \Phi_{tot}(r=b,\theta) = 0$$

$$\Rightarrow \sum_{l=0}^{\infty}\left[A_l b^l + \frac{q a^l}{b^{l+1}}\right] P_l(\cos\theta) = 0$$

$$\Rightarrow A_l = -\frac{q a^l}{b^{2l+1}}$$

$$\therefore \Phi_{tot}^{inside}(r,\theta) = \sum_{l=0}^{\infty}\left[-\frac{q a^l}{b^{2l+1}} r^l + \frac{q a^l}{r^{l+1}}\right] P_l(\cos\theta)$$

- Outside

$$\Phi_{sphere}(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

$$\Phi_q(r,\theta)\Big|_{r>a} = q\sum_{l=0}^{\infty} \frac{a^l}{r^{l+1}} P_l(\cos\theta)$$

$$\rightsquigarrow \Phi_{tot}(r=b,\theta) = 0$$

$$\sum_{l=0}^{\infty}\left(\frac{B_l}{b^{l+1}} + \frac{qa^l}{b^{l+1}}\right) P_l(\cos\theta) = 0$$

$$\Rightarrow \quad B_l = -qa^l$$

$$\Phi_{tot}^{outside}(r,\theta) = \sum_{l=0}^{\infty}\left[\frac{-qa^l}{r^{l+1}} + \frac{qa^l}{r^{l+1}}\right] P_l(\cos\theta) = 0$$

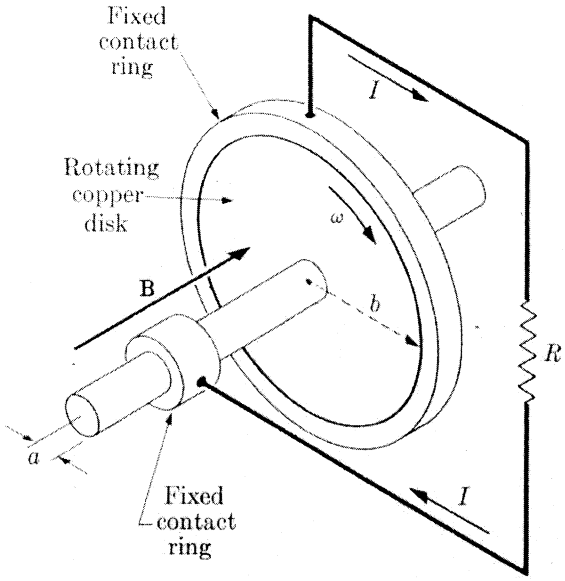
as we expect outside

(b)

$$\Phi_{sphere} = -\sum_{l=0}^{\infty} \frac{qa^l r^l}{b^{2l+1}} P_l(\cos\theta)$$

$$= \sum_{l=0}^{\infty} \frac{\left(-\frac{b}{a}\right) q r^l}{\left(\frac{b^2}{a}\right)^{l+1}} P_l(\cos\theta)$$

$$= q\left(-\frac{b}{a}\right) \sum_{l=0}^{\infty} \frac{r^l}{\left(\frac{b^2}{a}\right)^{l+1}} P_l(\cos\theta) = \frac{\overbrace{(-b/a)q}^{\text{imagecharge}}}{\underbrace{\left|\vec{r} - \frac{b^2}{a}\hat{z}\right|}_{\text{position}}}$$



4. The Homopolar Generator consists of a flat copper disk of radius b and thickness t, mounted on an axle of radius a, which mechanically rotates the disk with angular speed ω in the presence of an orthogonal magnetic induction $\mathbf{B}$. A stationary contact ring with inner radius b and negligible resistance surrounds the rotating disk making good electrical and frictionless contact with it. As shown in the figure, the closed electrical circuit consists of the disk and a load resistor R connected by wires between the axle and the stationary contact ring. (Assume the load resistor R is much greater than the resistance of the disk, the contact ring, and the wires.) A constant magnetic induction $\mathbf{B}$ perpendicular to the disk (parallel to the rotation axis) exists between the radii a and b and is zero elsewhere in the circuit.

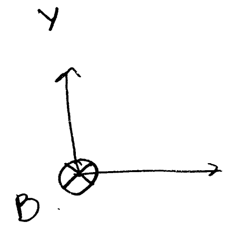
   (a) {4 pts} Find the current I that flows in the circuit as a function of B, a, b, ω, and R.

   (b) {2 pts}What is the magnitude of the current density $J(r)$ in the rotating disc.

   (c) {2 pts} What torque would you have to apply to the rotating wheel to keep ω from slowing down.

   (d) {2 pts} If σ is the conductivity of copper and t is the thickness of the disk, find the electrical resistance R_d of the disk between the radii a and b. Recall that the resistance of a small length $\Delta\ell$ of conducting material with cross sectional area A is $\Delta R = \Delta\ell/(\sigma A)$.

Prob 4 (Gaussian)



$\bar{B} = B(-\hat{z})$

$\bar{v} = v(-\hat{\phi})$

a) The electromotive force

$$\mathcal{E} = \int \bar{f}\cdot d\bar{l} = \int \left(\frac{\bar{v}}{c}\wedge\bar{B}\right)\cdot d\bar{l}$$

$$= \int_a^b \frac{\omega}{c} r B\, dr = \frac{\omega B}{c}\left[\frac{r^2}{2}\right]_a^b$$

$$= \frac{\omega B}{2c}(b^2 - a^2)$$

$$\bar{I} = \frac{\mathcal{E}}{R} = \frac{\omega B}{2RC}(b^2 - a^2)\,\hat{r}$$

b) $$\bar{J}(r) = \frac{dI}{da_\perp} = \frac{I}{2\pi r t}\hat{r} = \frac{\omega B (b^2 - a^2)}{4\pi R C t}\frac{1}{r}\hat{r}$$

c) The force per unit vol

$$F = \rho(\cancel{\vec{E}}^{\,0} + \frac{\bar{v}}{c}\wedge\bar{B}) = \rho\frac{\bar{v}}{c}\wedge\bar{B}$$

$$= \frac{\bar{J}\wedge\bar{B}}{c}$$

∴ The Force due to the current

$$d\bar{F} = \frac{1}{c}(\bar{J}\wedge\bar{B})$$

$$\bar{F} = \frac{1}{c}\int \frac{\omega B(b^2-a^2)}{4\pi R c t}\,\frac{1}{r}\,\hat{r}\wedge B(-\hat{z})\; r\,dr\,d\phi\,dz$$

$$\bar{F} = \hat{\phi}\,\frac{\omega B(b^2-a^2)}{4\pi R c^2 t}\int dr\,d\phi\,dz$$

$$\bar{F} = \frac{\omega B(b^2-a^2)}{4\pi R c^2 t}\,2\pi r t$$

Then the torque

$$d\bar{\tau} = \bar{r}\wedge d\bar{F}$$

$$\bar{\tau} = \int \bar{r}\wedge d\bar{F}\, d^3x$$

$$= \int r\hat{r}\wedge\left(\frac{\omega B(b^2-a^2)}{4\pi R c^2 t}\,\frac{1}{r}\,\hat{\phi}\right) d^3x$$

$$= \hat{z}\int \frac{\omega B(b^2-a^2)}{4\pi R c^2 t}\, r\,dr\,d\phi\,dz$$

$$= \hat{z}\,\frac{\omega B(b^2-a^2)}{4\pi R c^2 t}\,2\pi t\,(b^2-a^2)$$

$$= \hat{z}\,\frac{\omega B(b^2-a^2)^2}{2 R c^2}$$

Hence, torque applied by me,

$$\tau_{me} = -\hat{z}\,\frac{\omega B(b^2-a^2)}{2Rc^2}$$

d)

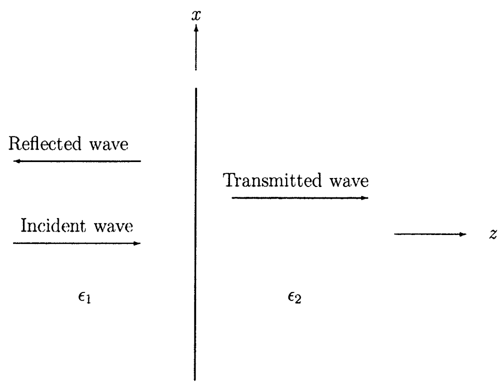
$$\Delta R = \frac{\Delta l}{\sigma A}$$

A ↳ cross-sectional area

$$A = 2\pi r t$$

$$R = \int dR = \int_a^b \frac{1}{r}\frac{dr}{2\pi\sigma t} = \frac{1}{2\pi\sigma t}\ln r\Big|_a^b$$

$$= \frac{1}{2\pi\sigma t}\ln\left(\frac{b}{a}\right)$$



5. A plane-polarized harmonic ($e^{-i\omega t}$) plane electromagnetic wave traveling to the right in a homogeneous dielectric medium described by an dielectric constant ϵ_1, strikes a second homogeneous dielectric material described by dielectric constant $\epsilon_2 > \epsilon_1$ (see the figure). Assume that both materials have the same magnetic permeability μ_0 and that the incidence angle is 0^o (i.e., the wave is traveling perpendicular to the junction). Assume the incoming wave is polarized in the $\hat{x}$ direction and that its electric field amplitude is E_0, i.e., assume the incoming electric field is the real part of

$$\mathbf{E} = E_0 \, e^{i(kz - wt)} \, \hat{x}.$$

(a) {3 pts} Give the magnetic induction $\mathbf{B}$ associated with the above incoming wave. Make sure your wave satisfies Maxwell's equations, e.g., give k as a function of ω, the direction of $\mathbf{B}$, and the amplitude of $\mathbf{B}$ as a function of E_0.

(b) {1 pts} Give similar expressions for the $\mathbf{E}$ and $\mathbf{B}$ components of the reflected and transmitted waves. Use E_0'' and E_0' for the respective amplitudes of reflected and transmitted waves.

(c) {2 pts} In general, what conditions must be satisfied at the junction between two materials by the electromagnetic fields $\mathbf{E}, \mathbf{B}, \mathbf{D}$, and $\mathbf{H}$, if Maxwell's equations are to be satisfied?

(d) {2 pts}Apply these junction conditions to the combined incoming, reflected, and transmitted wave to compute E_0'' and E_0' as functions of E_0 and the two dielectric constants ϵ_1 and ϵ_2.

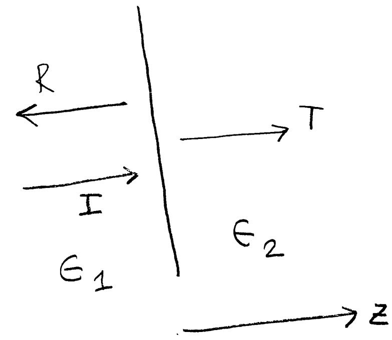
(e) {2 pts} Evaluate the time averages of the Poynting vectors of the incident, reflected, and transmitted waves. Recall that

$$\mathbf{S} \equiv \mathbf{E} \times \mathbf{H}, \qquad (SI)$$

$$\mathbf{S} \equiv \frac{1}{4\pi} \mathbf{E} \times \mathbf{H}. \qquad (Gaussian)$$

The sum of the magnitudes of the reflected and transmitted time averaged Poynting vectors should equal the magnitude of the incident wave's time averaged Poynting vector.

Prob 5 (Gaussian)



a) $\bar{E} = E_0 e^{i(kz-\omega t)}\,\hat{x}$

$\bar{B} = \sqrt{\mu_1 \epsilon_1}\;\hat{k} \wedge \bar{E}$ but $\mu_1 = 1$

$\Rightarrow \bar{B} = \sqrt{\epsilon_1}\; E_0 e^{i(kz-\omega t)}\,(\hat{z} \wedge \hat{x})$

$\Rightarrow \bar{B} = \sqrt{\epsilon_1}\; E_0 e^{i(kz-\omega t)}\,\hat{y}$

$\rightsquigarrow \nabla \wedge E = -\frac{1}{c}\frac{\partial \bar{B}}{\partial t}$

$$\Rightarrow \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \wedge \begin{pmatrix} E_0 e^{i(kz-\omega t)} \\ 0 \\ 0 \end{pmatrix} = -\frac{1}{c}(-i\omega)\sqrt{\epsilon_1}\,E_0 e^{i(kz-\omega t)}\,\hat{y}$$

$$\Rightarrow \begin{pmatrix} 0 \\ ik E_0 e^{i(kz-\omega t)} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{i\omega}{c}\sqrt{\epsilon_1}\,E_0 e^{i(kz-\omega t)} \\ 0 \end{pmatrix}$$

$\Rightarrow k = \frac{\omega}{c}\sqrt{\epsilon_1}$

$B_0 = \sqrt{\epsilon_1}\,E_0 \qquad \bar{B} = B\hat{y}$

$$\rightsquigarrow \quad \nabla \wedge \bar{H} = \frac{1}{c} \frac{\partial \bar{D}}{\partial t}$$

$$\rightsquigarrow \quad \frac{1}{\underbrace{\mu_1}_{=1}} \nabla \wedge \bar{B} = \frac{1}{c} \epsilon_1 \frac{\partial \bar{E}}{\partial t}$$

$$\Rightarrow -\sqrt{\epsilon_1}\, iK\, E_0\, e^{i(Kz-\omega t)}\, \hat{x} = \frac{1}{c} \epsilon_1 (-i\omega t)\, E_0\, e^{i(kz-\omega t)}\, \hat{x}$$

$$\Rightarrow \quad K = \frac{\omega}{c} \sqrt{\epsilon_1}$$

b) • Reflected Wave

$$\bar{E}_R = E_0'' e^{-i(kz+\omega t)}\, \hat{x}$$

$$\bar{B}_R = -\sqrt{\epsilon_1}\, E_0''\, e^{-i(Kz+\omega t)}\, \hat{y}$$

• Transmitted Wave

$$\bar{E}_T = E_0'\, e^{i(k_2 z-\omega t)}\, \hat{x}$$

$$\bar{B}_T = \sqrt{\epsilon_2}\, E_0'\, e^{i(k_2 z-\omega t)}\, \hat{y}$$

c) The electro magnetic boundary conditions

$$D^{\perp}_{above} - D^{\perp}_{below} = 4\pi \sigma_f \hat{n} = 0 \Rightarrow i)\ \epsilon_2 E_2^{\perp} - \epsilon_1 E_1^{\perp} = 0$$

ii) $E^{\parallel}_{above} - E^{\parallel}_{below} = 0$

iii) $B^{\perp}_{above} - B^{\perp}_{below} = 0$

$$H^{\parallel}_{above} - H^{\parallel}_{below} = 4\pi K_f \wedge \hat{n} = 0$$

iv) $\Rightarrow \frac{1}{\underbrace{\mu_0}_{=1}} \left(B^{\parallel}_{above} - B^{\parallel}_{below} \right) = 0$

d) i) $\epsilon_1 E_1^{\perp} = \epsilon_2 E_2^{\perp}$

$$\Rightarrow \epsilon_1 \left(E_0 e^{i(kz-\omega t)} + E_0'' e^{-i(kz+\omega t)} \right) = \epsilon_2 E_0' e^{i(k_2 z - \omega t)}$$

$$\Rightarrow \frac{\epsilon_1}{\epsilon_2} \left(E_0 e^{i(kz-\omega t)} + E_0'' e^{-i(kz+\omega t)} \right) = E_0' e^{i(k_2 z - \omega t)} \quad (1)$$

iii) $\sqrt{\epsilon_1} E_0 e^{i(kz-\omega t)} + \sqrt{\epsilon_1}\ E_0'' e^{-i(kz+\omega t)}$

$$= \sqrt{\epsilon_2}\, E_0' e^{i(k_2 z - \omega t)}$$

$$\Rightarrow \sqrt{\frac{\epsilon_1}{\epsilon_2}} \left(E_0 e^{i(kz-\omega t)} + E_0'' e^{-i(kz+\omega t)} \right) = E_0' e^{i(k_2 z - \omega t)} \quad (2)$$

(1) $\neq 2$

$$\Rightarrow \sqrt{\frac{\epsilon_1}{\epsilon_2}} \quad \cancel{\frac{E_0 e^{i(kz-\omega t)} + E_0'' e^{-i(kz+\omega t)}}{E_0'}}$$

• $\bar{E} = E\hat{x}$, $\bar{B} = B\hat{y}$ so $\bar{E}$ & $\bar{B}$ are both parallel to the plane of incidence

so i) & iii) are trivial

ii) $E''_{above}\Big|_{z=0} = E''_{below}\Big|_{z=0}$

$$E_0 e^{i(kz-\omega t)}\Big|_{z=0} + E_0'' e^{-i(kz+\omega t)}\Big|_{z=0} = E_0' e^{i(k_2 z-\omega t)}\Big|_{z=0}$$

$$\Rightarrow \quad E_0 + E_0'' = E_0' \quad \cdots (1)$$

iv) $B''_{above} = B''_{below}$

$$\Rightarrow \quad \sqrt{\epsilon_1}\left(E_0 - E_0''\right) = \sqrt{\epsilon_2}\, E_0' \quad \cdots (2)$$

(1) & (2)

$$2E_0 = \left(1 + \sqrt{\frac{\epsilon_2}{\epsilon_1}}\right) E_0' \Rightarrow E_0' = \frac{2E_0}{\left(1+\sqrt{\frac{\epsilon_2}{\epsilon_1}}\right)}$$

(1) & (2)

$$E_o\left(1-\sqrt{\frac{\epsilon_1}{\epsilon_2}}\right)+E_o''\left(1+\sqrt{\frac{\epsilon_1}{\epsilon_2}}\right)=0$$

$$\Rightarrow \quad E_o''=-\frac{\left(1-\sqrt{\frac{\epsilon_1}{\epsilon_2}}\right)}{\left(1+\sqrt{\frac{\epsilon_1}{\epsilon_2}}\right)}E_o$$

e) $$\bar{S}=\frac{c}{4\pi}\,\mathrm{Re}\left(\bar{E}\times\bar{H}\right)$$

$$\Rightarrow I_i=\langle S_\perp\rangle=\frac{c}{4\pi}\sqrt{\epsilon_1}\;E_o^2\left\langle\cos^2(kz-\omega t)\right\rangle\hat{z}=\frac{c}{8\pi}\sqrt{\epsilon_1}\,E_o^2$$

$$S_T=\frac{c}{4\pi}\left(\sqrt{\epsilon_2}\,E_o''^2\cos^2(kz+\omega t)\right)\quad(+\hat{z})$$

$$\Rightarrow I_T=\langle S_T\rangle=+\frac{c}{4\pi}\sqrt{\epsilon_2}\;\frac{4E_o^2}{\left(1+\sqrt{\frac{\epsilon_2}{\epsilon_1}}\right)^2}\left\langle\cos^2(\,)\right\rangle$$

$$=+\frac{c}{8\pi}\;\frac{4E_o^2\sqrt{\epsilon_2}}{\left(1+\sqrt{\frac{\epsilon_2}{\epsilon_1}}\right)^2}$$

$$\Rightarrow I_R=\langle S_R\rangle=\frac{c}{4\pi}\sqrt{\epsilon_1}\;E_o'^2\left\langle\cos^2(k_2z-\omega t)\right\rangle$$

$$=-\frac{c}{8\pi}\sqrt{\epsilon_1}\;\frac{\left(1-\sqrt{\frac{\epsilon_1}{\epsilon_2}}\right)^2E_o^2}{\left(1+\sqrt{\frac{\epsilon_1}{\epsilon_2}}\right)^2}$$

// $(a-b)^2 + 4ab = (a+b)^2$

$$I_i \overset{!}{=} I_R + I_T$$

$$= + \frac{c}{8\pi} \frac{4E_0^2\sqrt{\epsilon_2}}{\left(1+\sqrt{\frac{\epsilon_2}{\epsilon_1}}\right)^2} + \frac{c}{8\pi}\sqrt{\epsilon_1}\, E_0^2 \frac{\left(1-\sqrt{\frac{\epsilon_1}{\epsilon_2}}\right)^2}{\left(1+\sqrt{\frac{\epsilon_1}{\epsilon_2}}\right)^2}$$

$$= + \frac{cE_0^2}{8\pi}\left\{ \frac{+4\sqrt{\epsilon_2}\,\epsilon_1}{(\sqrt{\epsilon_1}+\sqrt{\epsilon_2})^2} + \frac{\sqrt{\epsilon_1}(\sqrt{\epsilon_2}-\sqrt{\epsilon_1})^2}{(\sqrt{\epsilon_1}+\sqrt{\epsilon_2})^2} \right\}$$

$$= \frac{cE_0^2}{8\pi}\sqrt{\epsilon_1}\left\{ \frac{4\sqrt{\epsilon_1}\sqrt{\epsilon_2} + (\sqrt{\epsilon_2}-\sqrt{\epsilon_1})^2}{(\sqrt{\epsilon_1}+\sqrt{\epsilon_2})^2} \right\}$$

$$= \frac{cE_0^2}{8\pi}\sqrt{\epsilon_1}$$

6. Maxwell's equations in 4 dimensions

   (a) {2 pts} Write the Maxwell equations in the absence of polarizable materials using 4-vector notation, making use of the field strength tensor $F_{\mu\nu}$.

   (b) {4 pts} Show that the equations of part (a) reduce to the usual form of Maxwell's equations in 3-vector notation.

   (c) {2 pts} The Lagrangian density of the EM field is given by

$$\mathcal{L} = -\frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu}, \qquad (SI)$$

   or

$$\mathcal{L} = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu}. \qquad (Gaussian)$$

   Recall that all repeated Greek indices are summed over 4-dimensions (1 time and 3 space). Show that the Lagrangian density is invariant under a gauge transformation $A_\mu \rightarrow A'_\mu = A_\mu + \partial_\mu \alpha(x)$, where α is an arbitrary function of spacetime $x \equiv (ct, \vec{x})$.

   (d) {2 pts} If we add an interaction term $\mathcal{L} \rightarrow \mathcal{L} + \Delta\mathcal{L}$ where

$$\Delta\mathcal{L} = j^\mu A_\mu, \qquad (SI)$$

   or

$$\Delta\mathcal{L} = \frac{1}{c} j^\mu A_\mu, \qquad (Gaussian)$$

   to the Lagrangian– where j^μ is some spatially bounded and conserved 4-current density– how does the action $I \equiv \int \mathcal{L} d^4 r$ change under a gauge transformation and do the resulting equations of motion change?

Prob 6 (Gaussian)

a) Homogeneous Maxwells Eqn

$$F_{[\alpha\beta,\gamma]} = 0$$

$$\Rightarrow \partial_\gamma F_{\alpha\beta} + F_{\alpha\beta\gamma} + F_{\beta\gamma\alpha} = 0$$

Inhomogeneous Maxwells Eqn

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu$$

b) did it before

c) $A_\mu \to A'_\mu = A_\mu + \partial_\mu \alpha(x)$

$$\mathcal{L} = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} = -\frac{1}{16\pi} g^{\mu\sigma} F_\sigma^{\ \nu} F_{\mu\nu}$$

$$= -\frac{1}{16\pi} \left(\partial^\mu A^\nu - \partial^\nu A^\mu\right)\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)$$

Now,

$$\partial^\mu A^\nu - \partial^\nu A^\mu = \partial^\mu\left(A'^\nu + \partial^\nu \alpha(x)\right) - \partial^\nu\left(A'^\mu + \partial^\mu \alpha(x)\right)$$

$$= \partial^\mu A'^\nu + \partial^\mu \partial^\nu \alpha(x) - \partial^\nu A'^\mu - \partial^\nu \partial^\mu \alpha(x)$$

$\because \partial^\mu \partial^\nu = \partial^\nu \partial^\mu$

$$= \partial^\mu A'^\nu - \partial^\nu A'^\mu + \underbrace{\left(\partial^\mu \partial^\nu \alpha(x) - \partial^\nu \partial^\mu \alpha(x)\right)}_{0}$$

Similarly

$$\partial_\mu A_\nu - \partial_\nu A_\mu$$

$$= \left(\partial_\mu A'_\nu - \partial_\nu A'_\mu\right) + \underbrace{\left(\partial_\mu \partial_\nu \alpha(x) - \partial_\nu \partial_\mu \alpha(x)\right)}_{0}$$

$$\begin{aligned}
\mathcal{L} = -\frac{1}{16\pi} \Big\{ & \left(\partial^\mu A'^\nu - \partial_\nu A'_\mu\right)\left(\partial_\mu A'_\nu - \partial_\nu A'_\mu\right) \\
& + \left(\partial^\mu A'^\nu - \partial_\nu A'_\mu\right)\left(\partial_\mu \partial_\nu \alpha(x) - \partial_\nu \partial_\mu \alpha(x)\right) \\
& + \left(\partial^\mu \partial^\nu \alpha(x) - \partial^\nu \partial^\mu \alpha(x)\right)\left(\partial_\mu A'_\nu - \partial_\nu A'_\mu\right) \\
& + \left(\partial^\mu \partial^\nu \underbrace{\alpha(x) - \partial^\nu \partial^\mu \alpha(x)\right)\left(\partial_\mu \partial_\nu \alpha(x) - \partial_\nu \partial_\mu \alpha(x)}_{=0}\right)
\end{aligned}$$

$$\mathcal{L}' = -\frac{1}{16} F'^{\mu\nu} F'_{\mu\nu}$$

$$\mathcal{L} = \mathcal{L}'$$

d) $\mathcal{L} \longrightarrow \mathcal{L} + \Delta\mathcal{L}$

$$\Delta\mathcal{L} = \frac{1}{c} j^{\mu} A_{\mu}$$

$$I = \int \mathcal{L}\, d^4 r = \int \mathcal{L}\, d^4 + \int \delta\mathcal{L}\, d^4 r$$

Under a Gauge transformation

$$A_{\mu} \longrightarrow A'_{\mu} + \partial_{\mu}\lambda$$

$$\delta I = \frac{1}{c} \int j^{\mu} (A'_{\mu} + \partial_{\mu}\lambda)\, d^4 r$$

$$= \frac{1}{c} \int j^{\mu} A'_{\mu}\, d^4 r + \frac{1}{c} \int \underbrace{j^{\mu} \partial_{\mu}}_{\partial_{\mu} j^{\mu} = 0} \lambda(x)\, d r^4$$

conservation of charge