Electrodynamics Qualifier Examination

January 9, 2008

General Instructions: In all cases, be sure to state your system of units. Show all your work, write only on one side of the designated paper, and if you get stuck on one part, assume a result and proceed onward. The points given for each part of each problem are indicated. Each problem carries equal weight.

1. This problem refers to macroscopic electrodynamics in a general medium. No specific relation is assumed between the fields $\mathbf{E}$ and $\mathbf{D}$, nor between $\mathbf{B}$ and $\mathbf{H}$.

a) 1 pt. Write the microscopic Maxwell equations containing as source the total charge density ρ and the total current density $\mathbf{j}$.

b) 1 pt. Show that the Maxwell equations of part (a) imply the conservation of the total charge Q.

c) 3 pt. Separate ρ into its two parts, the free charge density ρ_f and the (bound) polarization charge density ρ_p, and rewrite the Maxwell equation containing ρ to obtain the macroscopic form containing the displacement field $\mathbf{D}$ and ρ_f.

d) 3 pt. Separate $\mathbf{j}$ into its parts, the free current density $\mathbf{J}_f$, the polarization current density $\mathbf{J}_p$, and the magnetization current density $\mathbf{J}_m$, and rewrite the Maxwell equation containing $\mathbf{j}$ to obtain the form containing the magnetic field intensity $\mathbf{H}$, the displacement field $\mathbf{D}$, and the free current density $\mathbf{J}_f$.

e) 2 pt. Show that the free charge Q_f is conserved through the continuity equation containing ρ_f and $\mathbf{J}_f$, and then show that therefore the polarization charge Q_p is conserved.

Prob 1 (Gaussian)

a) $\nabla \cdot \bar{E} = 4\pi\rho$

$\nabla \wedge \bar{B} - \frac{1}{c}\frac{\partial \bar{E}}{\partial t} = \frac{4\pi}{c}\bar{J}$

b) $\underbrace{\nabla \cdot (\nabla \wedge \bar{B})}_{=0} - \frac{1}{c}\frac{\partial}{\partial t}(\nabla \cdot \bar{E}) = \frac{4\pi}{c}\nabla \cdot \bar{J}$

$\Rightarrow \quad \frac{\partial \rho}{\partial t} + \nabla \cdot \bar{J} = 0 \quad \Rightarrow$ conservation of total charge

c) $\nabla \cdot \bar{E} = 4\pi(\rho_f + \rho_p)$

$\Rightarrow \nabla \cdot \bar{E} - 4\pi\rho_p = 4\pi\rho_f$

but, $\rho_p = -\nabla \cdot \bar{P}$

$\Rightarrow \nabla \cdot \underbrace{(E + 4\pi\bar{P})}_{\bar{D}} = 4\pi\rho_f$

$\Rightarrow \nabla \cdot \bar{D} = 4\pi\rho_f$

d) $$\nabla \wedge \bar{B} - \frac{1}{c}\frac{\partial \bar{E}}{\partial t} = \frac{4\pi}{c}(J_f + J_p + J_m)$$

$$\Rightarrow \nabla \wedge \bar{B} - \frac{4\pi}{c} J_m - \frac{1}{c}\frac{\partial E}{\partial t} - \frac{4\pi}{c} J_p = \frac{4\pi}{c} J_f$$

but $J_m = +c\,\nabla \wedge \bar{M}$ & $J_p = \dfrac{\partial P}{\partial t}$

$$\Rightarrow \nabla \wedge (B - 4\pi\bar{M}) - \frac{1}{c}\frac{\partial}{\partial t}(E + 4\pi\bar{P}) = \frac{4\pi}{c} J_f$$

$$\Rightarrow \nabla \wedge \bar{H} - \frac{1}{c}\frac{\partial \bar{D}}{\partial t} = \frac{4\pi}{c}\bar{J}_f$$

e) $$\nabla \cdot (\nabla \wedge \bar{H}) - \frac{1}{c}\frac{\partial}{\partial t}(\nabla \cdot \bar{D}) = \frac{4\pi}{c}\nabla \cdot \bar{J}_f$$

$$\Rightarrow \frac{\partial \rho_f}{\partial t} + \bar{\nabla} \cdot \bar{J}_f = 0$$

$\therefore$ Q_f is conserved

we already showed $Q_{tot} = (Q_f + Q_p)$ is conserved

Hence Q_p has to be conserved

2. Consider an infinite slab of thickness d, carrying uniform charge density ρ, centered on the origin and extending in the x-y plane. Assume both the electric permittivity ε and the magnetic permeability μ have their vacuum values.

a) 2 pt. Find the electric field vector, $\mathbf{E}$, and magnetic flux density (magnetic induction), $\mathbf{B}$, everywhere. Do not just write the answer down, but in all cases clearly articulate your arguments and solution to receive credit.

For parts (b)–(e) consider an observer moving at velocity $\mathbf{v} = v_0\hat{\mathbf{x}}$. Do not assume that $v_0 \ll c$.

b) 1 pt. What is the current density, $\mathbf{J}'$, in the observer's frame of reference? [Hint: How does the charge density transform?]

c) 3 pt. Find the electric field vector, $\mathbf{E}'$, and magnetic flux density, $\mathbf{B}'$, everywhere in the observer's frame of reference. Do this by solving Ampére's and Gauss' law in the observer's frame.

d) 2 pt. Alternatively, obtain the same result by performing a Lorentz transformation on the fields found in part (a).

e) 2 pt. Show explicitly that $\mathbf{E} \cdot \mathbf{B}$ and $E^2 - B^2$ have the same value in both the rest frame of the slab and the observer's frame. Why is that? Is it possible to find a frame where $\mathbf{E} = \mathbf{0}$ and $\mathbf{B} \neq \mathbf{0}$?

3. Suppose there is a distribution of free charges (density ρ_f) with current density $\mathbf{J}_f$, in a medium with arbitrary $\mathbf{B}(\mathbf{H})$ and $\mathbf{D}(\mathbf{E})$, that is, there are two types of electric field, $\mathbf{D}, \mathbf{E}$, and two types of magnetic field, $\mathbf{B}, \mathbf{H}$.

a) 2 pt. Use the Lorentz force law to show that the rate at which the fields $\mathbf{E}$ and $\mathbf{B}$ do work on the charges in a volume V is given by

$$\frac{dW_f}{dt} = \int_V d^3x\, \mathbf{E} \cdot \mathbf{J}_f.$$

b) 5 pt. Use the result of part (a) and the macroscopic Maxwell equations to show Poynting's theorem,

$$\frac{dW_f}{dt} + \int_V d^3x\, \Upsilon + \oint_{\partial V} d\mathbf{a} \cdot \mathbf{S} = 0,$$

where ∂V is the surface enclosing the volume V, and $\mathbf{S}$ is the Poynting vector. Identify Υ, which in the absence of a medium is the rate of change of the electromagnetic energy density, $\partial u/\partial t$.

c) 3 pt. For a linear medium, with constant permittivity ε and constant permeability μ, state the relation between $\mathbf{D}$ and $\mathbf{E}$ and between $\mathbf{B}$ and $\mathbf{H}$, and show for such a medium the rate of change of electromagnetic energy density is given by

$$\Upsilon = \frac{\partial u}{\partial t}, \quad u = \frac{1}{2}\left(\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H}\right),$$

in either SI or Heaviside-Lorentz units.

Prob 3 (Gaussian)

a) Lorentz force for charge per unit volume

$$\bar{F} = \rho \left(\bar{E} + \frac{\bar{v}}{c} \wedge \bar{B} \right)$$

Now, the work done,

$$dW = \bar{F} \cdot d\bar{l} \quad \text{where } d\bar{l} = \bar{v}\, dt$$

$$= \left(\rho \bar{E} + \frac{\bar{v} \wedge \bar{B}}{c} \right) \cdot \bar{v}\, dt$$

as $(\bar{v} \wedge \bar{B}) \cdot \bar{v} = 0$

$$= \bar{E} \cdot (\rho \bar{v})\, dt$$

$$= \bar{E} \cdot \bar{J}_f \, dt$$

$$\Rightarrow \quad \frac{dW}{dt} = \bar{E} \cdot \bar{J}_f$$

Hence, Rate of work done in a vol V

$$\Rightarrow \frac{dW}{dt} = \int_V \bar{E} \cdot \bar{J}_f \, d^3x$$

b) $$\nabla \wedge \bar{H} - \frac{1}{c}\frac{\partial \bar{D}}{\partial t} = \frac{4\pi}{c}\bar{J}_f$$

$$\Rightarrow \quad \bar{J}_f = \frac{c}{4\pi}\left(\nabla \wedge \bar{H} - \frac{1}{c}\frac{\partial \bar{D}}{\partial t}\right)$$

Thus, $$\bar{E}\cdot\bar{J}_f = \frac{c}{4\pi}\,\bar{E}\cdot(\nabla\wedge\bar{H}) - \frac{1}{4\pi}\,\bar{E}\cdot\frac{\partial \bar{D}}{\partial t}$$

$$= \frac{c}{4\pi}\left\{-\nabla\cdot(\bar{E}\wedge\bar{H}) + \bar{H}\cdot(\nabla\wedge\bar{E})\right\} - \frac{1}{4\pi}\,\bar{E}\cdot\frac{\partial\bar{D}}{\partial t}$$

$$= \frac{c}{4\pi}\left(-\nabla\cdot(\bar{E}\wedge\bar{H})\right) + \frac{c}{4\pi}\left(-\frac{1}{c}\,\bar{H}\cdot\frac{\partial\bar{B}}{\partial t}\right) - \frac{1}{4\pi}\,\bar{E}\cdot\frac{\partial\bar{D}}{\partial t}$$

$$= \underbrace{\qquad\qquad} \quad -\frac{1}{4\pi}\left(\bar{H}\cdot\frac{\partial\bar{B}}{\partial t} + \bar{E}\cdot\frac{\partial\bar{D}}{\partial t}\right)$$

Now, $$-\int \nabla\cdot(\bar{E}\wedge\bar{H})\,d^3x = -\int_S (\bar{E}\wedge\bar{H})\cdot d\bar{s}$$

but $\bar{S} = \frac{c}{4\pi}(\bar{E}\wedge\bar{H})$

$$\therefore \frac{dW_f}{dt} + \oint_{\partial V} \bar{S}\cdot d\bar{a} + \frac{1}{4\pi}\int d^3x\,\underbrace{\left(\bar{H}\cdot\frac{\partial\bar{B}}{\partial t} + \bar{E}\cdot\frac{\partial\bar{D}}{\partial t}\right)}_{I}$$

c) Now,

$$\bar{D} = \epsilon \bar{E}$$

$$\bar{B} = \mu \bar{H}$$

thus,

$$\Upsilon = \frac{\partial u}{\partial t} = \frac{1}{4\pi}\left(\bar{H}\cdot\frac{\partial \bar{B}}{\partial t} + \bar{E}\cdot\frac{\partial \bar{D}}{\partial t}\right)$$

$$= \frac{\partial}{\partial t}(\bar{B}\cdot\bar{H}) - \bar{B}\cdot\frac{\partial \bar{H}}{\partial t} + \frac{\partial}{\partial t}(\bar{E}\cdot\bar{D})$$

$$- \frac{\partial \bar{E}}{\partial t}\cdot\bar{D}$$

$$\Rightarrow \quad \bar{H}\cdot\frac{\partial \bar{B}}{\partial t} = \mu\, \bar{H}\cdot\frac{\partial \bar{H}}{\partial t}$$

Now,

$$\frac{\partial}{\partial t}(\bar{H}\cdot\bar{H}) = \bar{H}\cdot\frac{\partial \bar{H}}{\partial t} + \bar{H}\cdot\frac{\partial \bar{H}}{\partial t}$$

$$\Rightarrow \quad \bar{H}\cdot\frac{\partial \bar{H}}{\partial t} = \frac{1}{2}(\bar{H}\cdot\bar{H})$$

Thus, $\bar{H}\cdot\dfrac{\partial \bar{B}}{\partial t} = \dfrac{1}{2}(\mu\bar{H}\cdot\bar{H}) = \dfrac{1}{2}(\bar{B}\cdot\bar{H})$

Similarly, $E\cdot\dfrac{\partial \bar{D}}{\partial t} = \epsilon\, \bar{E}\cdot\dfrac{\partial \bar{E}}{\partial t} = \dfrac{1}{2}\dfrac{\partial}{\partial t}(\epsilon\bar{E}\cdot E)$

$$= \frac{1}{2}\frac{\partial}{\partial t}(E\cdot\bar{D})$$

Hence,

$$\frac{\partial u}{\partial t} = \frac{1}{8\pi} \frac{\partial}{\partial t} \left(\bar{E} \cdot \bar{D} + \bar{B} \cdot \bar{H} \right)$$

$$\therefore \quad u = \frac{1}{8\pi} \left(\bar{E} \cdot \bar{D} + \bar{B} \cdot \bar{H} \right)$$

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4. Consider a monochromatic plane electromagnetic wave of frequency ω propagating in vacuum in the z direction and polarized in the x direction, which impinges upon a perfect conductor at $z = 0$, as shown in the figure. The incident electric field is

$$\mathbf{E}_I(z,t) = \hat{\mathbf{x}} E_{0I} e^{i(kz-\omega t)}.$$

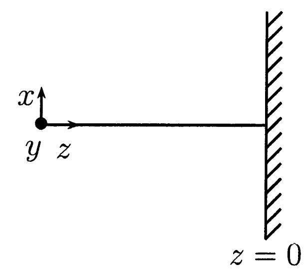

Figure 1: Plane wave normally incident on a perfectly conducting plane at $z = 0$.

a) 1 pt. Use Maxwell's equations to determine the relation between k and ω.

b) 1 pt. Use Maxwell's equations to determine the incident magnetic field, $\mathbf{B}_I(z,t)$.

c) 1 pt. What are the forms of the reflected wave $\mathbf{E}_R(z,t)$, $\mathbf{B}_R(z,t)$?

d) 2 pt. Apply the appropriate boundary conditions at the interface between the vacuum and the conductor to determine the reflected amplitudes E_{0R} and B_{0R} in terms of E_{0I}.

how does it work → e) 1 pt. What is the phase of the incident and reflected electric fields? Are they in phase or out of phase at $z = 0$?

*

f) 2 pt. What is the force exerted on the conducting surface by the reflection of the plane wave? Answer this question by computing the momentum transferred from the field to the conductor.

g) 2 pt. Answer the same question by computing the discontinuity of the normal-normal component of the stress tensor across the interface, ΔT_{zz}.

* K_f

* In general if $v_2 > v_1$ ($n_2 < n_1$) inphase

$v_1 > v_2$ ($n_1 < n_2$) out of phase

Prob 4 (Gaussian)

a) $$\bar{E}_I(z,t) = E_{0I}\, e^{i(kz-\omega t)}\,\hat{x}$$

i) $\nabla \wedge \bar{E} = -\frac{1}{c}\frac{\partial \bar{B}}{\partial t}$ iii) $\nabla \cdot \bar{E} = 0$

ii) $\nabla \wedge \bar{B} = \frac{1}{c}\frac{\partial \bar{E}}{\partial t}$ iv) $\nabla \cdot \bar{B} = 0$

i) $\Rightarrow \nabla \wedge (\bar{\nabla} \wedge \bar{E}) = -\frac{1}{c}\frac{\partial}{\partial t}(\nabla \wedge \bar{B})$

$$\Rightarrow \nabla(\underbrace{\nabla \cdot \bar{E}}_{=0}) - \nabla^2 \bar{E} = -\frac{1}{c^2}\frac{\partial^2 \bar{E}}{\partial t^2}$$

$$\Rightarrow \nabla^2 E = \frac{1}{c^2}\frac{\partial^2 \bar{E}}{\partial t^2}$$

$$\Rightarrow -k^2 E_{0I}\, e^{i(kz-\omega t)} = \frac{1}{c^2}(-i\omega)^2 E_{0I}\, e^{i(kz-\omega t}$$

$$\Rightarrow k^2 = \frac{\omega^2}{c^2}$$

ii) $\nabla \wedge \bar{B} = \frac{1}{c}\frac{\partial \bar{E}}{\partial t}$

$$\Rightarrow \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \wedge \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = \frac{1}{c}\begin{pmatrix} -i\omega E_{0I}\, e^{i(kz-\omega t)} \\ 0 \\ 0 \end{pmatrix}$$

$$\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = -\frac{i\omega}{c} E_0 e^{i(kz-\omega t)}$$

$$\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} = 0$$

$$\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = 0$$

$$\bar{B} = B\hat{y} \quad \Rightarrow \quad B_y =$$

then, $\longrightarrow$ Let's try Faraday's law

$$\leadsto \quad \nabla \wedge \bar{E} = -\frac{1}{c}\frac{\partial \bar{B}}{\partial t}$$

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \wedge \begin{pmatrix} E_{0I}\, e^{i(kz-\omega t)} \\ 0 \\ 0 \end{pmatrix} = \frac{1}{c}\frac{\partial \bar{B}}{\partial t}$$

$$\Rightarrow \begin{pmatrix} 0 \\ \frac{\partial}{\partial z} E_{0I} e^{i(kz-\omega t)} \\ \underbrace{\frac{\partial}{\partial y} E_{0I} e^{i(kz-\omega t)}}_{=0} \end{pmatrix} = \frac{1}{c}\frac{\partial \bar{B}}{\partial t}$$

$$\Rightarrow \hat{z}\; ik\, E_{0I}\, e^{i(kz-\omega t)} = \frac{1}{c}\frac{\partial \bar{B}}{\partial t}$$

$e^{i(kz-\omega t)}\, \partial t$

$\Rightarrow \frac{1}{c}\bar{B} = \hat{z}(-ik)\left(-\frac{1}{i\omega}\right) e^{i(kz-\omega t)} E_{0T}$ $\Rightarrow -i\omega dt = dx$

$=$

$\Rightarrow \bar{B}(z,t) = \hat{z}\frac{kc}{\omega} e^{i(kz-\omega t)} E_{0I}$

$\Rightarrow \bar{B}(z,t) = E_{0I}\, e^{i(kz-\omega t)}\hat{z}$

$\bar{k} = k(-\hat{z})$

c) $\bar{E}_R(z,t) = E_{0R}\, e^{-i(kz+\omega t)}\hat{x}$

$\bar{B}_R(z,t) = \hat{k}\wedge \bar{E}_R = -E_{0R}\, e^{-i(kz+\omega t)}\hat{y}$

d) i) $D^{\perp}_{above} - D^{\perp}_{below} = 4\pi\sigma_f\,\hat{n}$

where,
$\bar{D} = \epsilon\bar{E}$
$\bar{H} = \mu\bar{B}$

ii) $E^{\parallel}_{above} - E^{\parallel}_{below} = 0$

iii) $B^{\perp}_{above} - B^{\perp}_{below} = 0$

iv) $H^{\parallel}_{above} - H^{\parallel}_{below} = \frac{4\pi}{c}\bar{K}_f\wedge\hat{n}$

for perfect conductors, $\bar{K}_f = 0$

$E_{0T} = 0$

Since $\bar{E} = E\hat{x}$ & $\bar{B} = B\hat{y}$ i) & iii) are trivial

as both $\bar{E}$ & $\bar{B}$ are parallel to the plane of incidence

(ii) $E^{\parallel}_{below}\Big|_{z=0} = 0$

$$\Rightarrow \quad E_{0I}\, e^{i(kz-\omega t)}\Big|_{z=0} + E_{0R}\, e^{-i(kz+\omega t)}\Big|_{z=0} = 0$$

$$\Rightarrow \quad E_{0R} = -E_{0I}$$

$\because \bar{K}_f = 0 \quad B^{\perp}_{above} = 0$ & $\mu = 1$

iv) $B^{\parallel}_{below} = 0$

$$\Rightarrow \quad \bar{B}_{0I}\, e^{i(kz-\omega t)}\Big|_{z=0} + \bar{B}_{0R}\, e^{-i(kz+\omega t)}\Big|_{z=0} = 0$$

$$\Rightarrow \quad \bar{B}_{0R} = -\bar{B}_{0I} = -\bar{E}_{0I}$$

e) $$\bar{E}_I = E_{0I}\, e^{i(kz-\omega t)}\, \hat{x}$$

$$\bar{E}_R = -E_{0I}\, e^{i(-kz-\omega t)}\, \hat{x}$$

$$= E_{0I}\, e^{i(-kz-\omega t+\pi)}\, \hat{x}$$

since, $e^{i\pi} = \cos\pi + i\sin\pi = -1$

$\therefore$ phase diff is π so out of phase

f) The force on the conductor,

$$\bar{F} = \frac{d\bar{P}_{mech}}{dt} = -\frac{1}{4\pi c}\frac{d}{dt}\underbrace{\int_V (\bar{E}\wedge\bar{H})\,dv}_{\text{total momentum stored in em fields}} + \underbrace{\oint_S \overleftrightarrow{T}\cdot d\bar{a}}_{\text{momentum per unit time flowing in through the surface}}$$

Since there is no momentum flowing into the surface the 2nd integral is zero

Now, momentum density $\wp_{em} = \frac{1}{4\pi c}(\bar{E}\wedge\bar{H})$

the average momentum transferred to the system

$$\Delta p = \langle \wp_{em}\rangle \Delta t A$$

So, $F = \frac{\Delta P}{\Delta t} = \langle \wp_{em}\rangle A c$

$\Rightarrow$ force per unit area $= \frac{F}{A} = \langle \wp_{em}\rangle c$

$$\langle \wp_{em}\rangle = \frac{1}{4\pi c}\,c\,(-E_{oI}^2)\,\langle \cos^2(kz+\omega t)\rangle\,\hat{z}$$

$$= -\frac{E_{oI}^2}{8\pi}\hat{z}$$

g) $T_{\alpha\beta} \rightarrow$ is the force per unit area in αth direction acting on an element oriented in the βth direction

$$T_{\alpha\beta} = \frac{1}{4\pi}\left[E_\alpha D_\beta + H_\alpha B_\beta - \frac{1}{2}(\bar{E}\cdot\bar{D} + \bar{B}\cdot\bar{H})\,\delta_{\alpha\beta}\right]$$

$$T_{zz} = \frac{1}{4\pi}\left[-\frac{1}{2}(E^2+B^2)\right)$$

$$= -\frac{E_{0I}^2 \cos^2(kz-\omega t)}{4\pi}$$

f) The momentum transfered by the E-M wave

is

$$|\bar{P}_{em}| = \frac{1}{c^2}\int \vec{S}\, dv = \frac{1}{4\pi c}\int (\bar{E}\wedge\bar{H})\, dv$$

and the momentum density

$$\wp_{em} = \frac{1}{4\pi c}\mathrm{Re}(\bar{E}\wedge\bar{H})$$

$$= \frac{1}{4\pi c}\left(-E_{0I}^2\cos^2(kz+\omega t)\right)\hat{z}$$

$$= -\frac{E_{0I}^2\cos^2(kz+\omega t)}{4\pi c}\hat{z}$$

$\therefore$ Force per unit area

$$f = \frac{d\wp_{em}}{dt} = \hat{z}\,\frac{2E_{0I}^2\cos(kz+\omega t)\sin(kz+\omega t)\,\omega}{4\pi c}$$

$$f = \frac{\omega E_{0I}^2}{4\pi c}\sin(2(kz+\omega t))$$

9)
$$T_{\alpha\beta} = \frac{1}{4\pi}\left[E_\alpha D_\beta + H_\alpha B_\beta - \frac{1}{2}(\bar{E}\cdot\bar{D} + \bar{B}\cdot\bar{H})\delta_{\alpha\beta}\right]$$

$$(T_{zz})_I = \frac{1}{4\pi}\left[\cancel{E_z^2}^{\,0} + \cancel{B_z^2}^{\,0} - \frac{1}{2}(E^2+B^2)\right]$$

$$= \frac{1}{4\pi}\left\{-\frac{1}{2}\left[E_{0I}^2\cos^2(kz+\omega t) + E_{0I}^2\cos^2(kz+\omega t)\right]\right\}$$

$$= -\frac{1}{4\pi}\, 2E_{0I}^2\cos^2(kz+\omega t)$$

$$(T_{zz})_T = 0$$

$$\therefore\ \Delta T_{zz} = (T_{zz})_T - (T_{zz})_I = \frac{2E_{0I}^2\cos^2(kz+\omega t)}{4\pi}$$

$$\langle \Delta T_{zz}\rangle = \frac{E_{0I}^2}{4\pi}$$

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5. Consider a metallic conducting circular ring of inner and outer radii $r - \epsilon$ and r, respectively. Let the ring have thickness h perpendicular to the radius $\cancel{\ell}$. Assume $h, \epsilon \ll r$. Let ρ be the resistivity and μ be the mass density of the material from which the ring is made. The ring rotates (on frictionless bearings) with angular frequency ω about an axis along a diameter of the ring and thus has mechanical energy $K(\omega) = \frac{1}{2}I\omega^2$, where $I = \frac{1}{2}(\mu 2\pi r h \epsilon)r^2$ is the moment of inertia of the ring. (Note that $\mu 2\pi r h\epsilon$ is the mass of the ring.) There is a uniform magnetic field **B** perpendicular to the axis of rotation. See Figure.

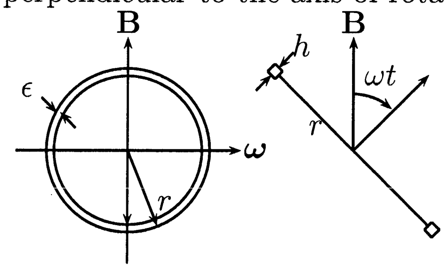


Figure 2: Top (left) and side (right) views of a conducting annulus, rotating with frequency ω about a diameter, with a magnetic field perpendicular to that axis of rotation.

a) 1 pt. Find the emf induced in the ring as a function of ω.

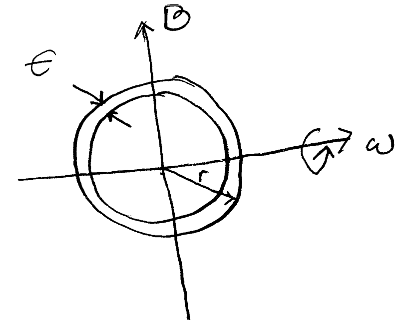
b) 3 pt. Find the average power dissipated by resistive heating in the ring as a function of ω, ρ, and B, and the relevant dimensional variables assuming that ω is constant.

c) 2 pt. If there is no external driving force keeping the ring rotating, the ring slows down due to this dissipation. Find the differential equation for $d\omega(t)/dt$ using $K(\omega)$ and the result of part (b). [The result of part (b) is correct if $d\omega/\omega \ll 1$.]

d) 2 pt. Solve this equation for ω as a function of time t in terms of the relevant variables such as μ, ρ, B, etc. Let ω_0 be the angular frequency at time $t = 0$.

e) 2 pt. If the annular ring were replaced by a solid disk of the same material with the same outer radius r and thickness h, would the disk take a longer time, a shorter time, or the same time as the annular ring to reach $\omega(t) = \omega_0/10$? You must support your answer with a physical argument to earn credit for this part.

Prob 5 (Gaussian)



a) the emf induced

$$\mathcal{E} = -\frac{1}{c}\frac{d}{dt}\int \bar{B}\cdot da\,\hat{\eta}$$

$$\bar{B}\cdot\hat{\eta} = B\cos\theta = B\cos\omega t$$

$$= -\frac{1}{c}\frac{d}{dt}\int \underline{B\,da}\cos\omega t$$

$\int B\, \underline{r}\, dr\, d\phi$

$$= -\frac{1}{c}\frac{1}{dt} B\pi r^2 \cos\omega t$$

$2\pi \int_{r-\epsilon}^{r} B\, r\, dr$

$$= +\frac{\omega}{c}\sin\omega t\, B\pi r^2$$

b)

$$\mathcal{E} = IR$$

$$\Rightarrow I = \frac{\omega}{Rc}\sin\omega t\, B(\pi r^2)$$

$$P = I^2 R$$

$$= \frac{\omega^2}{R^2c^2}\sin^2\omega t\, B^2\pi^2 r^4 R$$

$$\langle P\rangle = \frac{\omega^2 B^2(\pi^2 r^4)}{2c^2 R} \qquad R = \frac{\rho L}{A} = \frac{\rho 2\pi r}{h\epsilon}$$

$$= \frac{\omega^2 B^2}{2c^2}\frac{\pi^2 r^4 h\epsilon}{\rho 2\pi r} = \frac{\omega^2 B^2 h\epsilon\pi r^3}{4\rho c^2}$$

c)

$$\Rightarrow \quad \frac{dK(\omega)}{dt} + \underbrace{\frac{B^2 h \epsilon \pi r^3}{4\rho c^2}}_{\alpha} \omega^2 = 0$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{1}{2} I \omega^2\right) + \alpha \omega^2 = 0$$

$$\Rightarrow I \omega \frac{d\omega}{dt} + \alpha \omega^2 = 0$$

$$\Rightarrow \quad \frac{d\omega}{dt} + \frac{B^2 \cancel{h} \cancel{\epsilon} \cancel{\pi} \cancel{r^3}}{4\rho c^2 (\mu \cancel{\pi r h \epsilon}) \cancel{r^2}} \omega = 0$$

$$\Rightarrow \quad \frac{d\omega}{dt} + \frac{B^2}{4\mu\rho c^2} \omega = 0$$

d)

$$\Rightarrow \quad \frac{d\omega}{\omega} = -\frac{B^2}{4\mu\rho c^2} dt$$

$$\Rightarrow \int_{\omega_0}^{\omega} \frac{d\omega}{\omega} = -\underbrace{\frac{B^2}{4\mu\rho c^2}}_{\gamma} \int_0^t dt$$

$$\Rightarrow \ln\left(\frac{\omega}{\omega_0}\right) = \gamma t$$

$$\Rightarrow \quad \omega(t) = \omega_0 e^{-\gamma t}$$

e) Since, γ has the dimension of inverse time, the eqn is independent of the geometry of the object. No dependence in the eqn

So, the the time to reach $\frac{\omega_0}{10}$ would be same.

6. This problem gives a covariant form of Maxwell's equations appropriate for describing both classical and quantum radiation.

a) 2 pt. Write down Maxwell's equations in vacuum (that is, no dielectric or magnetic media are present, only free charges and currents) in covariant form in terms of the field-strength tensor $F^{\mu\nu}$ and the electric four-current density j^μ. *State your units, and the metric you are using.* What is the relation between $F^{\mu\nu}$ and the electric and magnetic field $\mathbf{E}$ and $\mathbf{B}$, and between the current j^μ and the electric charge density ρ and the electric current density $\mathbf{j}$.

b) 2 pt. Show that one set of Maxwell's equations is satisfied if $F^{\mu\nu}$ is derivable from a four-vector potential,

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu.$$

What is the relation between A^μ and the ordinary scalar and vector potentials ϕ and $\mathbf{A}$?

c) 1 pt. Suppose A^μ is given in the Lorenz gauge, where

$$\partial_\mu A^\mu = 0.$$

What is the equation relating A^μ to the current density j^μ?

d) 1 pt. Show that the equation for A^μ derived in part (c) implies the conservation of electric charge.

e) 2 pt. Now consider the Coulomb gauge condition given by $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$. Show that this leads to

$$\nabla^2 A^0 = -\frac{4\pi}{c} J^0 \quad (\nabla^2 \Phi = -4\pi\rho),$$

in Gaussian units.

f) 2 pt. What is the corresponding equation satisfied by the vector potential $\mathbf{A}$ in the Coulomb gauge?

Prob 6 (Gaussian)

a) $F_{[\mu\nu,\sigma]} = 0$

$$\left.\begin{aligned} &\Rightarrow \partial_\sigma F_{\mu\nu} + \partial_\mu F_{\nu\sigma} + \partial_\nu F_{\sigma\mu} = 0 \\ &\Rightarrow \partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu \end{aligned}\right\} \text{M.E}$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

$$E^i = -F^{0i}$$

$$F^{ij} = -\frac{1}{2}\epsilon_{ijk} B^k$$

b) $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$$\text{LHS} \Rightarrow \partial_\sigma(\partial_\mu A_\nu - \partial_\nu A_\mu) + \partial_\mu(\partial_\nu A_\sigma - \partial_\sigma A_\nu) + \partial_\nu(\partial_\sigma A_\mu - \partial_\mu A_\sigma) = 0$$

$$A^\mu = \begin{pmatrix} \Phi \\ \bar{A} \end{pmatrix}$$

c) $$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} j^\nu$$

$$\Rightarrow \partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) = \frac{4\pi}{c} J^\nu$$

$$\Rightarrow \partial_\mu \partial^\mu A^\nu - \partial^\nu \underbrace{\partial_\mu A^\mu}_{=0} = \frac{4\pi}{c} J^\nu$$

$$\Rightarrow \Box A^\nu = \frac{4\pi}{c} J^\nu$$

d) $$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\begin{pmatrix} \Phi \\ \bar{A} \end{pmatrix} = \frac{4\pi}{c}\begin{pmatrix} c\rho \\ J \end{pmatrix}$$

$$\Rightarrow \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} - \nabla^2 \bar{A} =$$

$$\partial_\nu (\partial_\mu \partial^\mu A^\nu) = \frac{4\pi}{c} \partial_\nu J^\nu$$

$$\Rightarrow \partial_\mu \partial^\mu \underbrace{(\partial_\nu A^\nu)}_{=0} = \frac{4\pi}{c} \partial_\nu J^\nu$$

$$\Rightarrow \partial_\nu J^\nu = 0$$ conservation of charge

e) $\quad \Box A^{\nu} = \frac{4\pi}{c} J^{\nu}$

$$\Rightarrow \left(\frac{1}{c^2}\frac{\partial}{\partial t^2} - \nabla^2\right) A^0 = \frac{4\pi}{c} c\rho$$

$$\Rightarrow \frac{1}{c^2}\frac{\partial^2 A^0}{\partial t^2} - \nabla^2 A^0 = 4\pi\rho$$

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^{\nu}$$

$$\Rightarrow \partial^2 A^{\nu} - \partial^{\nu} \underbrace{(\partial_\mu A^{\mu})} = \frac{4\pi}{c} J^{\nu}$$

$$-\partial^{\nu}(\partial_0 A^0 - \underbrace{\vec{\nabla}\cdot\vec{A}}_{0}) = \frac{4\pi}{c} J^{\nu}$$

$$\Rightarrow \partial^2 A^{\nu} - \partial^{\nu}\partial_0 A^0 = \frac{4\pi}{c} J^{\nu}$$

$$\Rightarrow \partial_0 \partial^0 A^0 - \nabla^2 A^0 - \partial^0 \partial_0 A^0 = \frac{4\pi}{c} J^0$$

$$\Rightarrow -\nabla^2 A^0 = \frac{4\pi}{c} J^0$$

$$\Rightarrow \nabla^2 A^0 = -\frac{4\pi}{c} J^0$$

f) $\cancel{\partial^2 A^\nu - \partial^\nu \partial_0 A^0 = \frac{4\pi}{c} J^\nu}$

$$\cancel{\Rightarrow \partial_0^2 A^i - \nabla^2 \bar{A} - \underbrace{\partial^i \partial_0 A^0}_{-\partial_0 \underbrace{\partial^i A^i}_{=0}} = \frac{4\pi}{c} J^i}$$

$$\cancel{\Rightarrow \frac{1}{c^2}\frac{\partial^2 \bar{A}}{\partial t^2} = \frac{4\pi}{c} \bar{J}}$$

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu$$

$$\Rightarrow \partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) = \frac{4\pi}{c} J^\nu$$

$$\Rightarrow \partial_\mu \partial^\mu A^\nu - \partial_\mu \partial^\nu A^\mu = \frac{4\pi}{c} J^\nu$$

$$\Rightarrow \partial_\mu \partial^\mu A^\nu - \partial^\nu (\partial_0 A^0 - \underbrace{\partial_i A^i}_{+\underbrace{\partial^i A^i}_{=0}}) = \frac{4\pi}{c} J^\nu$$

$$\Rightarrow \partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_0 A^0 = \frac{4\pi}{c} J^\nu$$

$$\Rightarrow \frac{1}{c^2}\frac{\partial^2 \bar{A}}{\partial t^2} - \nabla^2 \bar{A} + \bar{\nabla}\left(\frac{1}{c}\frac{\partial \Phi}{\partial t}\right) = \frac{4\pi}{c} \bar{J}$$