S·07

Electrodynamics Qualifier Examination

January 10, 2007

1. This problem deals with magnetostatics, described by a time-independent magnetic field, produced by a current density which is divergenceless,

$$\boldsymbol{\nabla} \cdot \mathbf{J} = 0.$$

Further, we are describing a nonmagnetic medium, so the relative permeability $\mu = 1$.

(a) [2 pts.] Show from Maxwell's equations that in the radiation gauge, where

$$\boldsymbol{\nabla} \cdot \mathbf{A} = 0,$$

the vector potential satisfies

$$-\nabla^2 \mathbf{A} = \frac{4\pi}{c} \mathbf{J},$$

in Gaussian units.

(b) [2 pts.] Show this equation may be solved as

$$\mathbf{A}(\mathbf{r}) = \frac{1}{c} \int d^3r' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}.$$

(c) [3 pts.] Now consider a sphere of radius a centered on the origin, which carries a total charge e distributed uniformly on its surface, and which is rotating with angular velocity $\boldsymbol{\omega}$, so that the velocity of a point $\mathbf{r}'$ on its surface is $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}'$. Use the result of part 1b as well as the Legendre expansion

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \sum_{l=0}^{\infty} \frac{r^l}{a^{l+1}} P_l(\cos \gamma),$$

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for $|\mathbf{r}| < a = |\mathbf{r}'|$, with γ being the angle between $\mathbf{r}$ and $\mathbf{r}'$, to compute both the vector potential, and the magnetic field *inside* the sphere.

(d) [3 pts.] Proceed in the same way to compute the vector potential and the magnetic field outside the sphere. What is the magnetic dipole moment of a rotating charged spherical shell?

Prob 1 (Gaussian)

a) $\nabla\wedge\bar{B} - \frac{1}{c}\frac{\partial\bar{E}}{\partial t} = \frac{4\pi}{c}\bar{J}$

$\nabla\wedge E = -\frac{1}{c}\frac{\partial\bar{B}}{\partial t}$

$\nabla\cdot\bar{E} = 4\pi\rho$

$\Rightarrow \bar{\nabla}\cdot\left(\frac{\partial E}{\partial t}\right) = 4\pi\frac{\partial\rho}{\partial t} = 4\pi\,(\nabla\cdot\bar{J}) = 0$

$\Rightarrow \frac{\partial E}{\partial t} = 0$

$\Rightarrow \underbrace{\nabla(\nabla\wedge\bar{B})}_{=0} - \frac{1}{c}\frac{\partial\bar{E}}{\partial t} = \underbrace{\nabla\cdot\bar{J}}_{=0}$

$\Rightarrow \frac{\partial\bar{E}}{\partial t} = 0$

$\Rightarrow -\nabla^2\bar{A} = \frac{4\pi}{c}\bar{J}$ ✓

$\Rightarrow \nabla\wedge(\nabla\wedge\bar{A}) - \frac{1}{c}\frac{\partial\bar{E}}{\partial t} = \frac{4\pi}{c}\bar{J}$

$\Rightarrow \nabla(\cancel{\nabla\cdot\bar{A}}^{\,0}) - \nabla^2 A = \frac{4\pi}{c}\bar{J}$

$\Rightarrow -\nabla^2 A = \frac{4\pi}{c}\bar{J}$

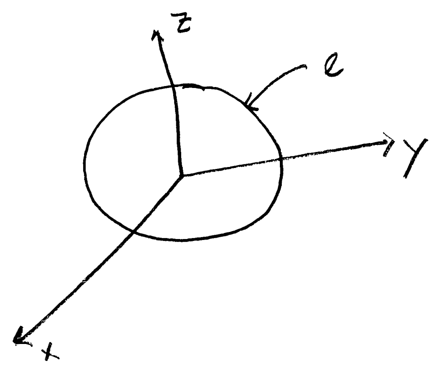
b) $$-\nabla^2 A = \frac{4\pi}{c}\bar{J}$$

$$-\nabla^2\left(\frac{1}{c}\int d^3r' \frac{\bar{J}(\bar{r}')}{|\bar{r}-\bar{r}'|}\right)$$

$$= -\frac{1}{c}\int d^3r' \frac{\nabla^2 \xrightarrow{0} \bar{J}(\bar{r}')}{|\bar{r}-\bar{r}'|} - \frac{1}{c}\int d^3r' \, \bar{J}(\bar{r}') \, \nabla^2\left(\frac{1}{|\bar{r}-\bar{r}'|}\right)$$

$$= -\frac{1}{c}\int d^3r' \, \bar{J}(\bar{r}')\left\{-4\pi\delta^3(\bar{r}-\bar{r}')\right\}$$

$$= \frac{4\pi}{c} J(r)$$



c) $\bar{v} \cancel{=} \cancel{\phi \times \bar{r}'}$

c) $$\bar{A}(\bar{r}) = \frac{1}{c}\int d^3r' \, \frac{\bar{J}(r')}{|\bar{r}-\bar{r}'|}$$

$$\sigma_0 = \frac{e}{4\pi a^2}$$

Now,

$$\bar{J}(r) = \sigma_0 v \, \delta(r-a)$$

$$\bar{J}(\bar{r}) = \sigma_0 (\bar{\omega} \times \bar{r}) \, \delta(r-a)$$

Thus, $$\bar{A}(\bar{r}) = \frac{1}{c}\int d^3r' \, \frac{\sigma_0(\bar{\omega}\times\bar{r}')}{|\bar{r}-\bar{r}'|}\delta(r'-a)$$

$$= \frac{\sigma_0 \bar{\omega}}{c} \times \int \frac{\bar{r}'}{|\bar{r}-\bar{r}'|}\delta(r'-a)\, d^3r'$$

$$|\bar{r}-\bar{r}'| = \sqrt{r^2+r'^2-2rr'\cos\gamma}$$

where $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$

Using spherical symmetry, we can choose

$$\bar{r} = r\hat{k}$$

now,

$$\bar{r}' = r'\hat{r}' = r'(\sin\theta'\cos\phi'\,\hat{x} + \sin\theta'\sin\phi'\,\hat{y} + \cos\theta'\,\hat{z})$$

$$\bar{A}(\bar{r}) = \frac{\sigma_0\,\bar{\omega}}{c} \times \int \frac{r'(\sin\theta'\cos\phi'\hat{x} + \sin\theta'\sin\phi'\hat{y} + \cos\theta'\hat{z})\,\overbrace{\delta(r'-a)}\,r'^2 dr' \sin\theta' d\theta' d\phi'}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta}}$$

but $\int_0^{2\pi} \cos\phi' d\phi' = \int_0^{2\pi} \sin\phi' d\phi' = 0$

$$\bar{A}(\bar{r}) = 2\pi \frac{\sigma_0}{c}\,\bar{\omega} \times \hat{z} \int \frac{r'\,\delta(r'-a)}{|\bar{r}-\bar{r}'|}\, r'^2 dr' \sin\theta' d\theta' \cos\theta'$$

Now,

$$\frac{1}{|\bar{r}-\bar{r}'|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\theta)$$

Then,

$$\int_0^{\pi} P_l(\cos\theta')\cos\theta' \sin\theta' d\theta'$$

$$= \int_0^{\pi} P_l(\cos\theta')\,P_1(\cos\theta')\sin\theta' d\theta'$$

$$= \frac{2}{3}$$

$$\bar{A}(\bar{r}) = \frac{4\pi\sigma_0}{3c}\,\bar{\omega}\times\hat{z}\int \frac{r_<}{r_>^2}\, r'^3\,\delta(r'-a)\,dr'$$

- <u>outside</u>

$$\bar{A}(\bar{r}) = \frac{4\pi\sigma_0\bar{\omega}}{3c}\times\hat{z}\int \frac{r'}{r^2}\, r'^3\,\delta(r'-a)\,dr'$$

$$= \frac{4\pi}{3}\frac{\sigma_0}{c}\,\frac{\bar{\omega}\times\hat{z}}{r^2}\int r'^4\,\delta(r'-a)\,dr'$$

$$= \frac{4\pi\sigma_0 a^4}{3c}\,\frac{\bar{\omega}\times\bar{r}}{r^3}$$

- Inside

$$\bar{A}(r) = \frac{4\pi\sigma_0\omega}{3c}\times\hat{z}\int \frac{r}{r'^2}\, r'^3\,\delta(r'-a)\,dr'$$

$$= \frac{4\pi\sigma_0 a}{3c}(\bar{\omega}\times\bar{r})$$

2. A transverse electric-magnetic (TEM) wave is transmitted using the *microstrip line* shown below, where the lower electrode is grounded. The microstrip is convenient for transmission of high-frequency signals on printed circuit boards. In practice, the width b of the strip is much larger than its distance h to the grounded plane, and edge effects are negligible. Let the current carried by each of the microstrip electrodes be I and the voltage between the electrodes be V.

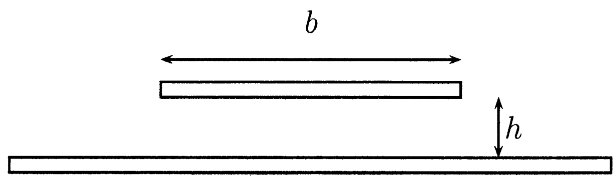


Figure 1: Cross section of microstrip line

(a) [2 pts.] Sketch the lines of **E** and **H** using arrows to show the directions of **E** and **H** at a given time. Show the direction of propagation of the wave.

(b) [3 pts.] Show that the instantaneous value of the transmitted power IV is equal to the Poynting vector integrated over the cross section bh.

(c) [3 pts.] Derive an expression for the characteristic impedance V/I of the microstrip line.

(d) [2 pts.] Show that the addition of a second grounded plane placed symmetrically with respect to the first reduces the characteristic impedance by a factor of two.

3. Light with wavenumber **k** and frequency ω is incident from vacuum normally along the z axis on a crystalline-plane parallel plate of thickness d. The crystal is cut so that a plane wave polarized along the x axis is refracted with index of refraction n_0 and one polarized along the y axis is refracted with index of refraction n_E.

   (a) [2 pts.] If the incident light is polarized at angle θ with respect to the x axis, write the form of the incident wave (using ω, θ, and space and time coordinates), assuming an incident amplitude of E_I.

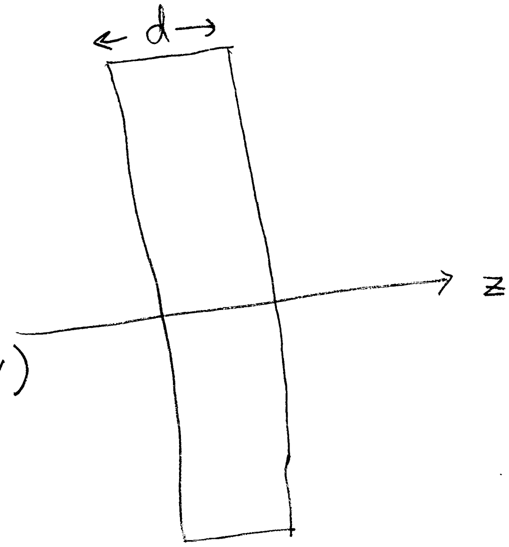
   (b) [2 pts.] Now write the form of the refracted wave inside the crystal (using ω, θ, n_0, n_E, and space and time coordinates) assuming that the refracted wave amplitude is E_T.

   (c) [6 pts.] Find the conditions of θ and d such that the refracted wave at d (the opposite side of the plate) is circularly polarized (either right or left).

problem

Prob 3 (Gaussian)

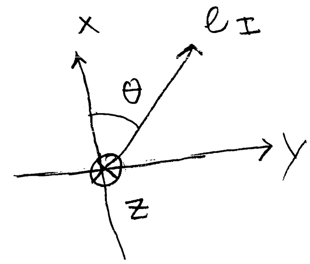
$$\vec{K} = K\hat{z}$$



a)

$$\hat{e}_I = \cos\theta\,\hat{x} + \sin\theta\,\hat{y}$$

$$\bar{E}_I = E_{oI}\, e^{i(kz-\omega t)}(\cos\theta\,\hat{x} + \sin\theta\,\hat{y})$$



$$\bar{B}_I = \sqrt{\mu_1\epsilon_1}\;\hat{k}\wedge\bar{E}_I$$

$$= E_{oI}\, e^{i(kz-\omega t)}\{\hat{z}\wedge(\cos\theta\,\hat{x} + \sin\theta\,\hat{y})\}$$

$$= E_{oI}\, e^{i(kz-\omega t)}(\cos\theta\,\hat{y} - \sin\theta\,\hat{x})$$

$$\bar{E}_R = E_{oR}\, e^{i(-kz-\omega t)}(\cos\theta\,\hat{x} + \sin\theta\,\hat{y})$$

$$\bar{B}_R = E_{oR}\, e^{i(-kz-\omega t)}(-\cos\theta\,\hat{y} + \sin\theta\,\hat{x})$$

b)

$$\bar{E}_T = (E_{oT})_x\, e^{i(k_o z-\omega t)}(\cos\theta\,\hat{x}) + (E_{oT})_y\, e^{i(k_E z-\omega t)}(\sin\theta\,\hat{y})$$

$$\bar{B}_T = \sqrt{\mu\epsilon}\;\hat{k}\wedge\bar{E}_T$$

$$= n_o\,(E_{oT})_x\, e^{i(k_o z-\omega t)}(\cos\theta\,\hat{y})$$

$$+\; n_E\,(E_{oT})_y\, e^{i(k_E z-\omega t)}(-\sin\theta\,\hat{x})$$

Now, the boundary conditions are

$\mu = 1$

i) $(E_{OI} + E_{OR})_z = (E_{OT})_z$

ii) $(E_{OI} + E_{OR})_{x,y} = (E_{OT})_{x,y}$

iii) $(\hat{k}_I \wedge E_{OI} + \hat{k}_R \wedge E_{OR})_z = \sqrt{\epsilon}\,(\hat{k}_T \wedge E_{OT})_z$

iv) $(\hat{k}_I \wedge E_{OI} + \hat{k}_R \wedge E_{OR})_{x,y} = \sqrt{\epsilon}\,(\underline{\hat{k}_T \wedge E_{OT}})$

i) & iii) are trivial no z-components of $\bar{E}$ & $\bar{B}$

ii) $(E_{OI} \cos\theta + E_{OR} \cos\theta)_x = (E_{OT})_x \cos\theta$

$(E_{OI} \sin\theta + E_{OR} \sin\theta)_y = (E_{OT})_y \sin\theta$

iv) $(-E_{OI} \sin\theta + E_{OR} \sin\theta)_x = -n_E (E_{OT})_y \sin\theta$

$(E_{OI} \cos\theta - E_{OR} \cos\theta)_y = n_o (E_{OT})_x \cos\theta$

/* $\rightsquigarrow (E_{OT})_x \cos\theta + n_o n_E (E_{OT})_y \sin\theta = E_{OI} (\sin\theta + \cos\theta)$

$\rightsquigarrow (E_{OT})_y \sin\theta + n_o \cancel{n_E} (E_{OT})_x \cos\theta = E_{OI} (\sin\theta + \cos\theta)$

$\rightsquigarrow \quad 2E_{0I}\cos\theta = (E_{0T})_x \cos\theta\,(1+n_o)$

$$\rightsquigarrow \quad (E_{0T})_x = \frac{2E_{0I}}{1+n_o}$$

$\rightsquigarrow \quad 2E_{0I}\sin\theta = (E_{0T})_y \sin\theta\,(1+n_E)$

$$\Rightarrow \quad (E_{0T})_y = \frac{2E_{0I}}{1+n_E}$$

So, For circular polarization

$$(E_{0T})_x \cos\theta = (E_{0T})_y \sin\theta$$

$$\Rightarrow \quad \frac{1+n_E}{1+n_o} = \tan\theta$$

$$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{1+n_E}{1+n_o}\right)$$

the phase for X & Y components should differ by $\pi/2$

thus, $k_o(z+d) - \omega t \sim k_E(z+d) + \omega t = \pi/2$

$$\Rightarrow (k_o \sim k_E)z + (k_o \sim k_E)d = \pi/2$$

$$\Rightarrow d = \frac{\pi/2 - (k_o \sim k_E)}{(k_o \sim k_E)}$$

✓

4. Two equal and opposite point charges, $\pm Q$, are located on the z axis at $\mathbf{r} = (0, 0, \pm b)$, as shown in the figure.

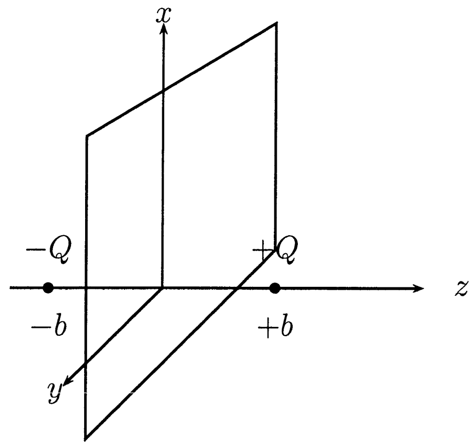


Figure 2: Equal and opposite charges lying on the z axis at $z = \pm b$. Shown also is the imaginary plane $z = 0$.

(a) [4 pts.] Compute Maxwell's stress tensor T^{ij} on the $z = 0$ plane, where

$$T^{ij} = \frac{1}{4\pi}\left(E^i E^j - \frac{1}{2}\delta^{ij} E^2\right).$$

(b) [4 pts.] Evaluate the integral over the $z = 0$ plane

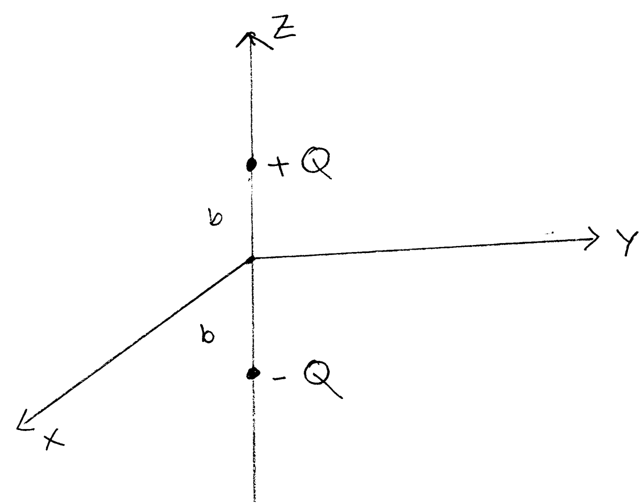
$$\int_{z=0} dA_j T^{ij} = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, T^{iz}\bigg|_{z=0},$$

using cylindrical polar coordinates.

(c) [2 pts.] Is your answer related to the force on the negative charge? Why?

Prob 4 (Gaussian)

$$\bar{E} = \frac{Q}{r^3}\bar{r}$$



a) $\bar{r} = x\hat{x} + y\hat{y} + z\hat{z}$

$$\bar{r}' = b'\hat{z}$$

$$\bar{E}_+ = \frac{Q(x\hat{x} + y\hat{y} - b'\hat{z})}{(x^2+y^2+b^2)^{3/2}}$$

$$\bar{E}_- = \frac{-Q(x\hat{x} + y\hat{y} + b'\hat{z})}{(x^2+y^2+b^2)^{3/2}}$$

$$E = \bar{E}_+ + \bar{E}_- = \frac{-2Qb'}{(x^2+y^2+b^2)^{3/2}}\hat{z}$$

$$T^{ij} = \frac{1}{4\pi}\left(E^iE^j - \frac{1}{2}\delta^{ij}E^2\right)$$

the only non-zero elements are the diagonal elements

$$T_{xx} = -\frac{2Q^2b'^2}{(x^2+y^2+b^2)^3} = T_{yy}$$

$$T_{zz} = \frac{4Q^2b'^2}{(x^2+y^2+b^2)^3} - \frac{2Q^2b'^2}{(x^2+y^2+b^2)^3}$$

$$= \frac{2Q^2b^2}{(x^2+y^2+b^2)^3}$$

$$T^{ij} = \frac{1}{4\pi}\frac{2Q^2b'^2}{(x^2+y^2+b^2)^3}\begin{pmatrix} -1 & & 0 \\ & -1 & \\ 0 & & +1 \end{pmatrix}$$

b)

$$\rightarrow \int_{z=0} dA_j T^{ij}$$

$\hat{r} = \cos\theta\,\hat{x} + \sin\theta\,\hat{y}$
$\hat{\theta} = -\sin\theta\,\hat{x} + \cos\theta\,\hat{y}$

$$= \oint \sum_{\beta} T_{\alpha\beta} n_\beta \, da$$

$$d\bar{a} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} r\,dr\,d\theta$$

$$= \oint T_{zz} n_z \; r\,dr\,d\theta$$

$$= \int -\frac{2Q^2 b^2}{(r^2+b^2)^3} \, r\,dr\,d\theta$$

$$= -Q^2 b^2 \int_0^\infty -\frac{r\,dr}{(r^2+b^2)^3}$$

$r^2+b^2 = x$
$\Rightarrow 2r\,dr = dx$
$\Rightarrow r\,dr = \frac{1}{2}\,dx$

$$= -\;Q^2 b^2 \frac{1}{2} \int_{b^2}^{\infty} x^{-3}\,dx$$

$r = 0;\; x = b^2$
$r = \infty,\; x = \infty$

$$= -\;Q^2 b^2 \frac{1}{2} \left[-\frac{1}{2} x^{-2} \right]_{b^2}^{\infty}$$

$$= \frac{1}{4} Q^2 b^2 \left[\frac{1}{\infty} - \frac{1}{b^4} \right]$$

$$= -\frac{Q^2}{4b^2}$$

c) $\bar{F} = \dfrac{Q(-Q)}{(2b)^2} = -\dfrac{Q^2}{4b^2}$

So the forces are related.

We are basically finding the force betn the charges

If we find a stress tensor, we need a plane to find the force.

5. A very long (infinitely long) and very thin uniformly charged rod is parallel to the z axis and located at $x = a$ and $y = 0$. An infinite grounded conducting sheet is located in the plane $x = 0$, as shown in the figure.

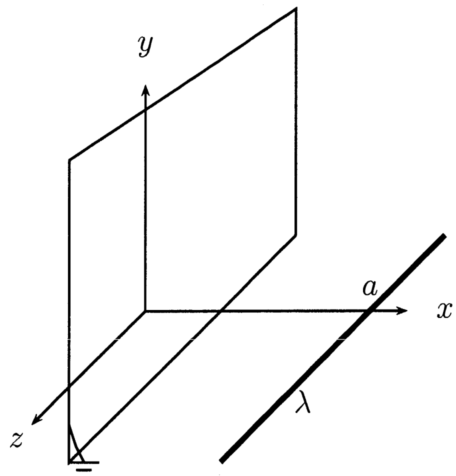


Figure 3: Charged rod, carrying uniform charge of density λ per unit length, parallel to, and a distance a from a grounded plane at $x = 0$.

(a) [2 pts.] What are the boundary conditions on the electric field at the conducting plane?

(b) [5 pts.] Compute the charge density (charge per unit area) on the conducting sheet $\sigma(y)$.

(c) [3 pts.] Evaluate the integral

$$-\lambda = \int_{-\infty}^{\infty} dy\, \sigma(y)$$

to check your value of σ.

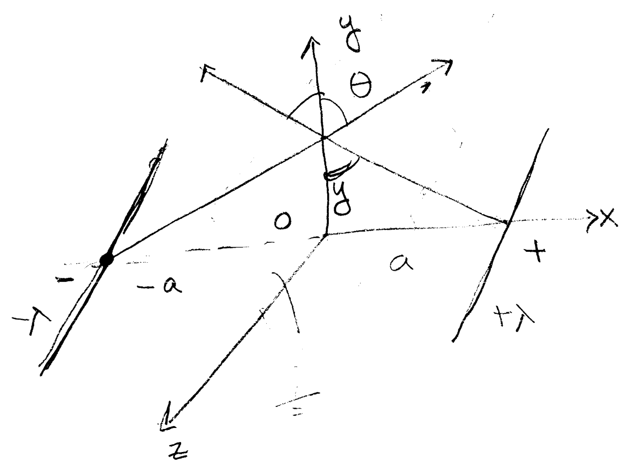
Prob 5 (Gaussian)

(a) $E^{\perp}_{above} - E^{\perp}_{below} = 4\pi\sigma$

$E^{\parallel}_{above} - E^{\parallel}_{below} = 0$

b) * Note the charge distribution extends to infinity, so the coulomb law is no good

→ Consider a image line charge at $x = -a$ of charge density $-\lambda$



Using Gauss's law of $+\lambda$

$E\, 2\pi r_+ L = 4\pi\lambda L$

$\Rightarrow \bar{E}_{above} = \frac{2\lambda}{r_+}(-\hat{r}) = E^{\perp}_{above}$

$E\, 2\pi r_- L = -\frac{2\lambda}{r_-}(-\hat{r}) = \frac{2\lambda}{r_-}\hat{r} = \bar{E}_{below},$

$\Rightarrow E_{\perp}^{above} - E_{\perp}^{below} = 4\pi\sigma$

$\Rightarrow +\frac{2\lambda}{r}\sin\theta - \frac{2\lambda}{r}\cos\theta - \frac{2\lambda}{r}\sin\theta - \frac{2\lambda}{r}\cos\theta = 4\pi\sigma$

$\Rightarrow -\frac{4\lambda}{r}\cos\theta = 4\pi\sigma \Rightarrow -\frac{4\lambda}{r}\frac{a}{r} = 4\pi\sigma$

$$\Rightarrow \sigma = -\frac{\lambda a}{\pi(y^2+a^2)}$$

c) $$-\lambda = \int_{-\infty}^{+\infty} dy\, \sigma(y)$$

$y = a\cot\theta$

$dy = -a\csc^2\theta\, d\theta$

$y = \infty,\ \theta = 0$

$y = -\infty,\ \theta = \pi$

$$\Rightarrow -\int \frac{\lambda a}{\pi(y^2+a^2)}\, dy$$

$$= -\frac{\lambda a}{\pi}\int_0^{\pi} \frac{+a\csc^2\theta\, d\theta}{a^2\csc^2\theta\, d\theta}$$

$$= -\lambda \checkmark$$

6. This problem describes a crossover network for a high-fidelity speaker system. Here we write $V = V_0 e^{-i\omega t}, I = I_0 e^{-i\omega t}$, where I_0 and V_0 are complex constants, and $\omega = 2\pi f$ is the angular frequency. We relate the voltage and current by $V = ZI$, where Z is the complex impedance.

   (a) [1 pt.] Starting from the basic definition of capacitance, $C = Q/V$, and inductance, $L = V/(dI/dt)$, derive the impedances Z_C and Z_L of a capacitor and an inductor, respectively.

   The figure shows the crossover network for a high-fidelity speaker system, which has a high-frequency "tweeter" and a low-frequency "woofer," each of which have a voice-coil with the same resistance R. The two capacitors each have a capacitance C and the two inductors have an inductance L.

   Using the standard equations for impedances in series and parallel:

   (b) [2 pts.] Find Z_W, the impedance (between points "a" and "c" of the R, L, C network of the "woofer" network.

   (c) [2 pts.] Find Z_T, the impedance (between points "b" and "d" of the R, L, C network of the "tweeter" network.

   (d) [3 pts.] In the high-fidelity speaker system, points "a" and "b" are connected together to one of the two output terminals of the amplifier, while points "c" and "d" are connected to the other output terminal. If $L = 2CR^2$, find Z_{total}, the impedance of the high-fidelity speaker system, that is, the impedance between point "a-b" and point "c-d." Comment on the frequency dependence of this impedance.

   (e) [2 pts.] The crossover frequency ω_c of the speaker system is defined as the frequency at which the "tweeter" and the "woofer" each receive half of the power delivered by the amplifier. For a given value of R and ω_c, determine L and C.