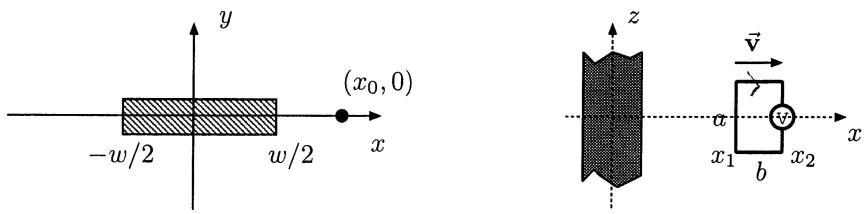
Electricity & Magnetism Qualifier

PSfrag replacements

For each problem, state what system of units you are using.

1. The diagram shows an end view of an infinitely long current sheet of width w and negligible thickness. A total current I travels in the $+z$ direction (out of the paper for the figure on the left) and is distributed uniformly over the width w, from $x = -w/2$ to $x = +w/2$.



   (a) [1 points] What is the direction of the magnetic field B at the point $x = x_0$, $y = 0$?

   (b) [4 points] What is the magnitude of B at $x = x_0$, $y = 0$?

   (c) [4 points] Now consider a rectangular loop of wire (dimensions $a \times b$) traveling to the right along the x-axis as shown in the figure with velocity $\vec{\mathbf{v}}$. The loop of wire is in the x-z plane, which is the same plane as the current sheet. Calculate the induced EMF on the rectangular loop (reading on the voltmeter). *Note that* $w/2 < x_1 < x_2$

   (d) [1 point] Which terminal of the voltmeter (upper or lower) is positive?

2. A cylindrical capacitor is formed from two coaxial conductors of length L. The inner conductor has an outer radius A while the outer cylinder is hollow and has inner radius B, which is slightly greater than A. [Assume $B - A \ll L$ so that end (fringe) effects can be neglected.]

   (a) [3 points] Derive the capacitance C_0 of the capacitor in terms A, B, and L.

   A hollow cylinder of dielectric material with inner radius A and outer radius B and length $X < L$ is now inserted between the two conductors. It has dielectric constant $K = \epsilon/\epsilon_0$ where ϵ is the permittivity of the material and ϵ_0 is the permittivity of free space.

   (b) [2 points] Calculate the new capacitance C of the capacitor in terms of C_0, L, X, and K.

   (c) [2 points] A battery with voltage V_0 is connected across the capacitor and then disconnected leaving the capacitor charged. Find σ_{free}, the free surface charge density at $r = A$ on the portion of the inner conducting cylinder covered by the dielectric material. Find σ_{bound}, the polarization charge density (or bound charge density) on the inner surface of the dielectric material at $r = A$.

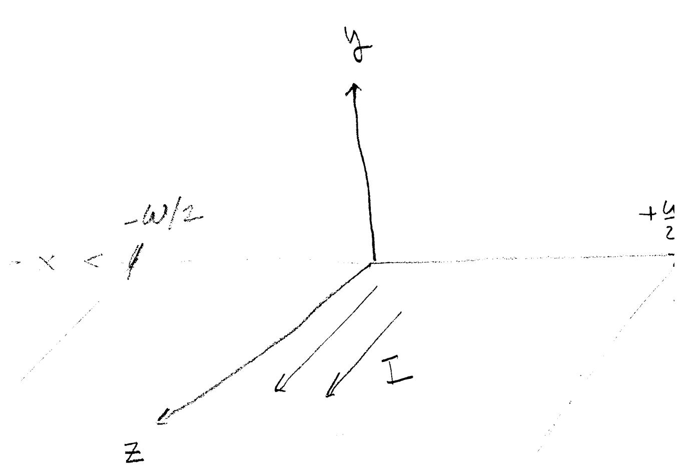
   (d) [3 points] With the capacitor still charged (as in part d), find the force F required to pull the dielectric cylinder from the capacitor when it is inserted a depth X between the two conductors and comment on how this force varies with X.

1. Problem 1 (Gaussian)

a) Using right hand rule,

$$\bar{B}(x_0, 0) = B\hat{y}$$

b) $\bar{B} = \frac{1}{c}\int \frac{\bar{K}\wedge(\bar{r}-\bar{r}')}{|\bar{r}-\bar{r}'|^3}\, d^2x'$



$$\bar{K} = \frac{\bar{I}}{w} = \frac{I\hat{z}}{w}$$

$$\bar{r} = x_0\hat{x}$$

$$\bar{r}' = x'\hat{x} + z'\hat{z}$$

$$\bar{r}-\bar{r}' = (x_0 - x')\,\hat{x} - z'\hat{z}$$

$$\bar{B} = \frac{I}{wc}\int \frac{\hat{z}\wedge[(x_0-x')\hat{x} - z'\hat{z}]}{[(x_0-x')^2 + z'^2]^{3/2}}\, dx'dz'$$

$$= \frac{I}{wc}\int_{-w/2}^{+w/2}\int_{-\infty}^{+\infty} \frac{(x_0-x')\,\hat{y}}{[(x_0-x')^2 + z'^2]^{3/2}}\, dx'dz'$$

$z' = (x_0 - x)\cot\theta$

$dz' = -(x_0 - x)\csc^2\theta$

$$= \frac{I}{wc}\int_{-w/2}^{+w/2}\int_0^{\pi} \frac{(x_0-x')^2\csc^2\theta}{(x_0-x')^3\csc^3\theta}\, d\theta'\, dx'\,\hat{y}$$

$$= \frac{I}{wc}\int_{-w/2}^{+w/2}\frac{1}{(x_0-x')}\int_0^{\pi}\sin\theta\, d\theta'\, dx\;\hat{y}$$

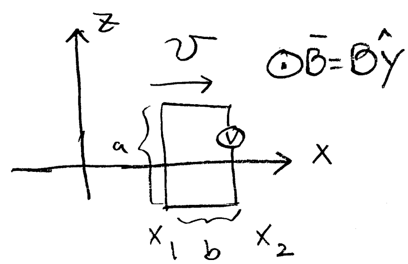
$$\bar{B} = \hat{y}\frac{2I}{\omega c}\int_{-\omega/2}^{+\omega/2} \frac{dx'}{(x_0 - x')}$$

$$x_0 - x' = \alpha$$
$$\Rightarrow -dx' = d\alpha$$
$$\alpha = x_0 - \frac{\omega}{2}$$
$$= x_0 + \frac{\omega}{2}$$

$$= \hat{y}\,\frac{2I}{\omega c}\int_{x_0 - \frac{\omega}{2}}^{x_0 + \frac{\omega}{2}} \frac{d\alpha}{\alpha}$$

$$= \hat{y}\,\frac{2I}{\omega c}\,\ln\alpha\Big|_{x_0 - \frac{\omega}{2}}^{x_0 + \frac{\omega}{2}}$$

$$\bar{B} = \hat{y}\,\frac{2I}{\omega c}\ln\left(\frac{x_0 + \omega/2}{x_0 - \omega/2}\right)$$



c) $\bar{v} = v\hat{x} \qquad \bar{B} = B\hat{y}$

$$\mathcal{E} = -\frac{1}{c}\frac{d\Phi}{dt} = -\frac{1}{c}\frac{d}{dt}\int \bar{B}\cdot d\bar{a} \qquad \frac{\omega}{2} < x_1 < x_2$$

$$= -\frac{1}{c}\frac{d}{dt}\int_{x_1 = x_{10} + vt}^{x_2 = x_{20} + vt} \frac{2Ia}{\omega c}\ln\left(\frac{x + \omega/2}{x - \omega/2}\right)dx$$

$$= -\frac{1}{c}\frac{d}{dt}\int_{x_1}^{x_2} \frac{2Ia}{\omega c}\ln\left(\frac{x + \frac{\omega}{2}}{x - \frac{\omega}{2}}\right)v\,dt$$

$$= -\frac{2Ia}{\omega c^2}\left[\ln\left(\frac{x + \frac{\omega}{2}}{x - \frac{\omega}{2}}\right)\right]_{x_1}^{x_2}$$

$$x_{20} - x_{10} = x_2 - x_1 = b$$

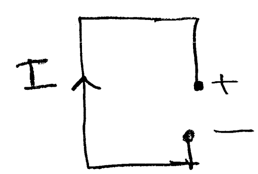
$$\mathcal{E} = -\frac{1}{c}\frac{d}{dt}\int_{x_1 = x_{10}+vt}^{x_2 = x_{20}+vt} \frac{2Ia}{wc} \ln\left(\frac{x+w/2}{x-w/2}\right) dx$$

$$= -\frac{1}{c}\frac{d}{dt}\int_{x_1}^{x_2} \frac{2Ia}{wc} \ln\left(\frac{x+w/2}{x-w/2}\right) v\,dt$$

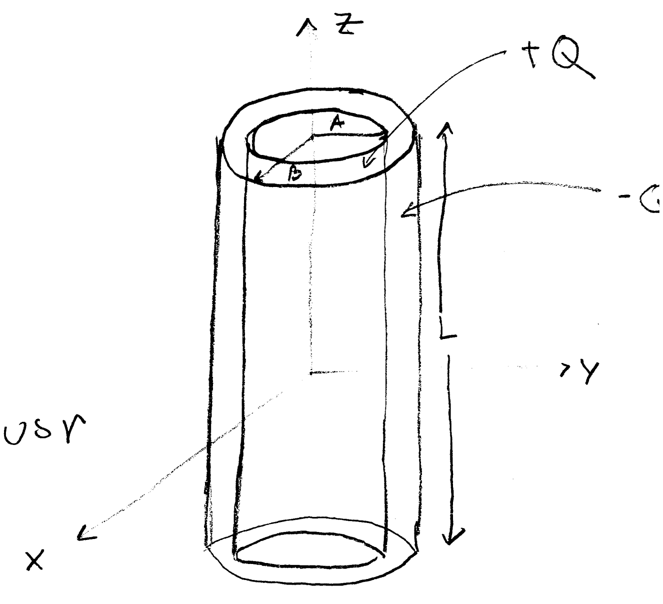
$$= -\frac{2Iav}{wc^2}\left[\ln\left\{\left(\frac{x_{20}+vt+\frac{w}{2}}{x_{20}+vt-w/2}\right)\frac{(x_{10}+vt-w/2)}{(x_{10}+vt+w/2)}\right\}\right]$$

$$= -\frac{2Iav}{wc^2}\left[\ln\frac{(b+x_{10}+vt+\frac{w}{2})(x_{10}+vt-w/2)}{(b+x_{10}+vt-w/2)(x_{10}+vt+w/2)}\right]$$

d) As we move away the B-flux reduces, so according to Lenz's law the current will flow in the direction to oppose the change i.e. clockwise then the upper terminal is positive.



Prob 2 (Gaussian)



a) Capacitance, $C = \frac{Q}{V}$

→ Choose Gaussian surface of radius r

where, $A < r < B$

→ Put +Q charge on the inner cylinder & -Q charge on the out cylinder

→ Assume the volume charge density & the surface density uniform

$$\int \bar{E} \cdot d\bar{a} = 4\pi Q_{enc}$$

$$\Rightarrow \quad E \, 2\pi r L = 4\pi Q$$

$$\Rightarrow \quad \bar{E} = \frac{2Q}{rL} \hat{r}$$

$$V = -\int_A^B \bar{E} \cdot d\bar{r} = -\int_A^B \frac{2Q}{rL} dr = -\frac{2Q}{L} \ln \Big|_A^B$$

$$= \frac{2Q}{L} \ln(A/B) \quad \Rightarrow \quad C_0 = \frac{L}{2 \ln(A/B)}$$

b) $$\int_0^x \bar{D}\cdot d\bar{a} + \int_x^L \bar{E}\cdot d\bar{a} = 4\pi Q_{fenc}$$

$$\Rightarrow \epsilon E\, 2\pi r x + E\, 2\pi r (L-x) = 4\pi Q$$

$$\Rightarrow E\, 2\pi r(\epsilon x + L - x) = 4\pi Q$$

$$\Rightarrow E = \frac{2Q}{\{(\epsilon-1)x + L\} r}$$

$$V = -\int_A^B \bar{E}\cdot d\bar{r} = -\frac{2Q}{\{(\epsilon-1)x+L\}} \ln r \Big|_A^B$$

$$= \frac{2Q}{\{(\epsilon-1)x+L\}} \ln\left(\frac{A}{B}\right)$$

$$\Rightarrow C = \frac{Q}{V} = \frac{\{(\epsilon-1)x+L\}}{2\ln(A/B)}$$

$$= C_0 + \frac{(\epsilon-1)x}{2\ln(A/B)}$$

c) If Q charge is on the inner conductor

$$\sigma_f = \frac{Q}{2\pi A L} \times 2\pi A x = \frac{Qx}{L}$$

$$\bar{D} \cdot 2\pi A x = 4\pi \frac{Qx}{L} 2\pi A x$$

$$C = \frac{Q}{V}$$

$$\Rightarrow \quad \bar{D} = \frac{4\pi Q x}{L} \hat{r}$$

$$\frac{V_0 (\epsilon - 1) x}{2 \ln (A/B)} = Q$$

$$\therefore \quad \bar{E} = \frac{4\pi Q x}{\epsilon L} \hat{r}$$

$$\Rightarrow \quad \bar{D} = \bar{E} + 4\pi \bar{P}$$

$$\Rightarrow \quad \bar{P} = \frac{1}{4\pi} (\bar{D} - \bar{E}) = \frac{Qx}{L}\left(1 - \frac{1}{\epsilon}\right) \hat{r}$$

$$\sigma_b = \bar{P} . \hat{n} = \frac{Qx}{L}\left(1 - \frac{1}{\epsilon}\right)$$

$$= \frac{V_0 \epsilon \left(1 - \frac{1}{\epsilon}\right)^2 x^2}{2 L \ln (A/B)}$$

d) Keeping the charge const.

$$W = \frac{1}{2}\frac{Q^2}{C}$$

$$F = -\frac{dW}{dx} = \frac{Q^2}{2C^2}\frac{dC}{dx}$$

$$\frac{dc}{dx} = \frac{(\epsilon-1)}{2\ln(A/B)}$$

$$F = \frac{Q^2}{2C^2}\frac{(\epsilon-1)}{2\ln(A/B)}$$

$$= \frac{1}{2}Q^2 \frac{4\{\ln(A/B)\}^2}{\{(\epsilon-1)x+L\}^2}\frac{(\epsilon-1)}{2\ln(A/B)}$$

$$= \frac{Q^2\ln(A/B)}{(\epsilon-1)\{(\epsilon-1)x+L\}}$$

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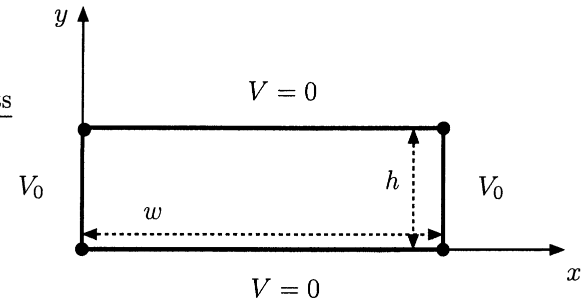
3. A very long channel (aligned with the z-axis) is made of four conducting plates insulated from each other. The top and bottom plates are grounded while the two side plates are held at a potential V_0 (see figure). Find the electric potential, $V(x, y)$, at any point inside the channel.

PSfrag replacements



(a) [2 points] Using Laplace's equation and the method of separation of variables, find the differential equations for $X(x)$ and $Y(y)$ assuming $V(x, y) = X(x)Y(y)$.

(b) [2 points] Determine the general solutions of the differential equations in part (a).

(c) [3 points] Find the particular solutions in part (b) that satisfy the boundary conditions. Express your answer as a Fourier series with arbitrary coefficients.

(d) [3 points] Use Fourier's method to evaluate the coefficients in part (c).

4. Electromagnetic plane waves

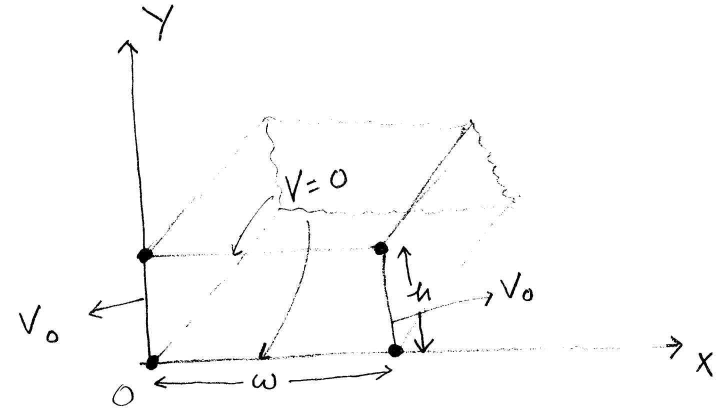
(a) [1 point] Write down real expressions for the $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ fields of a plane wave linearly polarized in the $\hat{\mathbf{x}}$ direction. (Take E_0 and B_0 as the amplitudes.)

(b) [2 points] For the wave of part a, determine the instantaneous and time-averaged energy density, u and $< u >$. Also determine the instantaneous Poynting vector and its time average $\vec{\mathbf{S}}$ and $< \vec{\mathbf{S}} >$.

(c) [1 point] A circular loop of wire can be used to detect electromagnetic waves. Suppose a 100 MHz FM station radiates 50 kW uniformly in all directions. What is the wavelength of the radiation.

(d) [6 points] What is the maximum rms voltage induced in a loop of radius 0.3 m at a distance of 10^5 m from the station in part c.

Prob 3 (Gaussian)



a) $V(x,y) = X(x)\,Y(y)$

$$\frac{d^2V(x,y)}{dx^2} + \frac{d^2V(x,y)}{dy^2} = 0$$

$$\Rightarrow Y(y)\frac{d^2X(x)}{dx^2} + X(x)\frac{d^2Y(y)}{dy^2} = 0$$

$$\Rightarrow \frac{1}{X(x)}\frac{d^2X(x)}{dx^2} + \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} = 0$$

they each has to be equal to some const

Let, $\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} = -k^2$

then, $\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = k^2$

b) $\frac{d^2Y(y)}{dy^2} = -k^2 Y(y) \Rightarrow Y(y) = Ae^{iky} + Be^{-iky}$

$= A\cos ky + B\sin ky$

$\frac{d^2X(x)}{dx^2} = k^2 X(x) \Rightarrow X(x) = Ce^{kx} + De^{-kx}$

c) the BC's:

i) $x = 0;\quad V = V_o$

ii) $x = w,\quad V = V_o$

iii) $y = 0,\quad V = 0$

iv) $y = h,\quad V = 0$

$\rightsquigarrow$ iii) $0 = A$

$\rightsquigarrow$ iv) $0 = B \sin kh$

$$\Rightarrow \quad \sin n\pi = \sin kh$$

$$\Rightarrow \quad k = \frac{n\pi}{h}, \qquad n = 0, 1, 2, \ldots$$

$\rightsquigarrow$ i) $V_o = C + D$

$\rightsquigarrow$ ii) $V_o = Ce^{kw} + De^{-kw}$

$$C + D = Ce^{kw} + De^{-kw}$$

$$\Rightarrow \quad C(1 - e^{kw}) = D(e^{-kw} - 1)$$

$$\Rightarrow \quad C = De^{-kw}$$

d) $X(x) = D e^{-kw} e^{kx} + D e^{-kx}$

$\rightsquigarrow X(x) = D\left(e^{-kw} e^{kx} + e^{-kx}\right)$

$\rightsquigarrow Y(y) = B \sin ky$ where, $k = \frac{n\pi}{h}$

$$V(x,y) = \sum_{n=0}^{\infty} \overbrace{B_n \sin\left(\frac{n\pi}{h}y\right) D_n}^{C_n} \left(e^{-\frac{n\pi w}{h}} e^{\frac{n\pi x}{h}} + e^{-\frac{n\pi}{h}x}\right)$$

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d) $$V(x,y) = \sum_{n=0}^{\infty} \sin\left(\frac{n\pi}{h}y\right)\left(C_n e^{\frac{n\pi}{h}x} + D_n e^{-\frac{n\pi}{h}x}\right)$$

$\rightsquigarrow$ i) $$V_0 = \sum_{n=0}^{\infty} \sin\left(\frac{n\pi}{h}y\right)(C_n + D_n)$$

$$\Rightarrow \int_0^h V_0 \sin\left(\frac{n'\pi}{h}y\right)dy = \sum_{n=0}^{\infty} (C_n + D_n) \int_0^h \sin\left(\frac{n\pi}{h}y\right)\sin\left(\frac{n'\pi y}{h}\right)dy$$

$$n = n' \quad \int_0^h \sin\left(\frac{n\pi}{h}y\right)\sin\left(\frac{n'\pi}{h}y\right)dy = h/2$$

$$\Rightarrow \quad C_n + D_n = \frac{2V_0}{h}\int_0^h \sin\left(\frac{n\pi}{h}y\right)dy$$

$$\Rightarrow \quad C_n + D_n = \frac{2V_0}{h} \int_0^h \sin\left(\frac{n\pi}{h} y\right) dy$$

$$= \frac{2V_0}{h}\left(-\frac{h}{n\pi}\right) \cos\left(\frac{n\pi y}{h}\right)\Big|_0^h$$

$$= -\frac{2V_0}{n\pi}\left[\cos(n\pi) - 1\right]$$

$$= \begin{cases} \frac{4V_0}{n\pi}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

BC iv) $\Rightarrow$

$$V_0 = \sum_{n=0}^{\infty} \sin\left(\frac{n\pi}{h} y\right)\left(C_n e^{\frac{n\pi}{h} w} + D_n e^{-\frac{n\pi}{h} w}\right)$$

$$\Rightarrow \quad \frac{2V_0}{h} \int_0^h \sin\left(\frac{n\pi}{h} y\right) dy = C_n e^{\frac{n\pi w}{h}} + D_n e^{-\frac{n\pi}{h} w} \quad ; n = n'$$

$$\Rightarrow \begin{cases} \frac{4V_0}{n\pi} = C_n e^{\frac{n\pi w}{h}} + D_n e^{-\frac{n\pi w}{h}}, & n \text{ odd} \\ 0 = C_n \, 2 \sinh\left(\frac{n\pi w}{h}\right), & n \text{ even} \end{cases}$$

$\Rightarrow n = 0$ $\quad$ $\sinh 0 = 0$

$$C_n = \frac{4V_0}{n\pi} - D_n$$

$$\Rightarrow \frac{4V_0}{n\pi} = \left(\frac{4V_0}{n\pi} - D_n\right) e^{\frac{n\pi w}{h}} + D_n e^{-\frac{n\pi w}{h}}$$

$$\Rightarrow \frac{4V_0}{n\pi}\left(e^{\frac{n\pi w}{h}} - 1\right) = D_n \left(e^{\frac{n\pi w}{h}} - e^{-\frac{n\pi w}{h}}\right)$$

$$= D_n \, 2 \sinh\left(\frac{n\pi w}{h}\right)$$

$$\Rightarrow D_n = \frac{2V_0 \left(e^{n\pi w/h} - 1\right)}{n\pi \sinh\left(\frac{n\pi w}{h}\right)}$$

$$\Rightarrow C_n = \frac{2V_0}{n\pi}\left\{2 - \frac{e^{n\pi w/h} - 1}{\sinh\left(\frac{n\pi w}{h}\right)}\right\} =$$

$$V(x,y) = \frac{2V_0}{\pi} \sum_{\substack{n=0 \\ n \text{ odd}}}^{\infty} \sin\left(\frac{n\pi}{h} y\right) \left\{ \left(2 - \frac{e^{n\pi w/h} - 1}{\sinh\left(\frac{n\pi w}{h}\right)}\right) e^{\frac{n\pi}{h} x} + \frac{e^{n\pi w/h} - 1}{\sinh\left(\frac{n\pi w}{h}\right)} e^{-\frac{n\pi}{h} x} \right\}$$

Prob 4 (Gaussian)

$\bar{E} = E\hat{x}$ let, $\vec{K} = K\hat{z}$

then $\bar{B} = B\hat{y}$

a) $\bar{E} = E_0 e^{i(\bar{K}\cdot\bar{r} - \omega t)} \hat{x} = E_0 \cos(Kz - \omega t)\, \hat{x}$

$\bar{B} = \hat{K} \wedge \bar{E}_0 = B_0 \cos(Kz - \omega t)\, \hat{y}$

Assuming the plane wave is in vacuum

b) $u = \frac{1}{8\pi}(E^2 + B^2)$

$= \frac{1}{8\pi}(E_0^2 + B_0^2)\cos^2(kz - \omega t)$

$\langle u \rangle = \frac{1}{16\pi}(E_0^2 + B_0^2)$

$\bar{S} = \frac{c}{4\pi}(\bar{E} \wedge \bar{B}) = \frac{c}{4\pi} E_0 B_0 \cos^2(Kz - \omega t)\, \hat{z}$

$\langle \bar{S} \rangle = \frac{c}{8\pi} E_0 B_0$

c) $\nu = 10^2\ MHz = 10^8\ Hz$ $\nu = \frac{1}{T}$

$\langle S \rangle = 50 \times 10^3\ W$ $\lambda = cT$

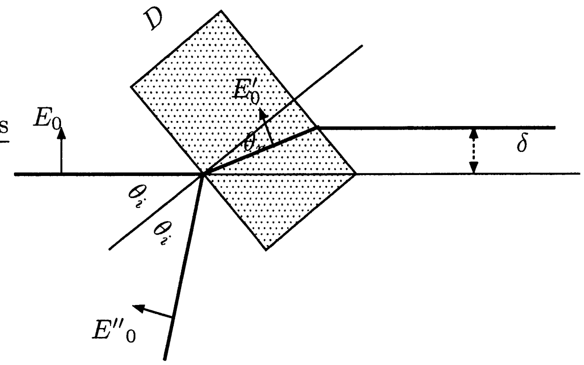
$= \frac{c}{\nu}$

$$\lambda = \frac{3 \times 10^8\ m\,s^{-1}}{10^8\ s^{-1}} = 3\ m$$

d) $V_{rms} =$

5-06

5. Consider light from a laser hitting a flat piece of glass as shown. The laser is linearly polarized with the plane of polarization lying in the scattering plane as shown. The angle of incidence is adjusted until the reflection is zero (i.e., until θ_i=Brewster's angle.) In this problem you will find the displacement δ in terms of the thickness of the glass D and the index of refraction n. You may assume that the index of refraction of air is 1 and that $\mu_{air} = \mu_{glass} = \mu_o$. $= 1$



(a) [4 points] In the absence of free charge, Maxwell's equations in integral form are given by

i. $\oint \vec{D} \cdot d\vec{A} = 0$

ii. $\oint \vec{E} \cdot d\vec{\ell} = -\frac{d}{dt} \int \vec{B} \cdot d\vec{A}$

iii. $\oint \vec{B} \cdot d\vec{A} = 0$

iv. $\oint \vec{H} \cdot d\vec{\ell} = \frac{d}{dt} \int \vec{D} \cdot d\vec{A}$

Each of these equations imposes a constraints on the variables θ_i, θ_r, E'_o, E_o, E''_o and n. Derive the constraints and specifically show that Eqs. (i) and (iv) lead to the same constraint on the electric field amplitudes. (ε and μ do not have to appear because $\mu = \mu_o$. Recall $\varepsilon\mu = 1/v^2 = n^2/c^2$)

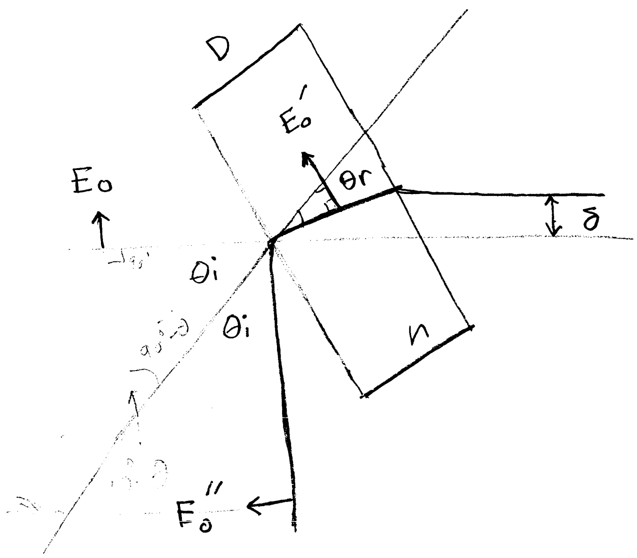
(b) [2 points] Using the constraints derived above, in addition to Snell's law, derive an expression for E''_0 in terms of n, θ_i, and E_0.

(c) [2 points] Use your answer to part b to obtain the value of θ_i for which E''_o goes to zero (i.e., find Brewster's angle.)

(d) [2 points] Derive an expression for δ in terms of n and D at the Brewster angle?

Prob 5 (Gaussian)

a)



i) $\oint \bar{D}\cdot d\bar{a} = 0$

$\Rightarrow D^{\perp}_{above} - D^{\perp}_{below} = 0$

$\Rightarrow \epsilon_1 E_1^{\perp} = \epsilon_2 E_2^{\perp}$

$n = \sqrt{\epsilon \underset{1}{\mu}} = \sqrt{\epsilon}$

$\rightsquigarrow (E_0 + E_0'')^{\perp} = n^2 E_0'^{\perp}$

sin	all +
tan	tanc

$\Rightarrow E_0 \sin(90° - \theta_i) + E_0'' \sin(90° - \theta_i)$

$= n^2 E_0' \sin(90° - \theta_r)$

$\Rightarrow E_0 \cos\theta_i + E_0'' \cos\theta_i = n^2 E_0' \cos\theta_r$ (i)

ii) $\oint \bar{E}\cdot d\ell = -\frac{1}{c}\frac{d}{dt}\int \bar{B}\cdot d\bar{a}$

$\Rightarrow E^{\parallel}_{above} - E^{\parallel}_{below} = 0$

$\Rightarrow (E_0 + E_0'')^{\parallel} = E_0'^{\parallel} \Rightarrow (E_0 + E_0'')\sin\theta_i = E_0' \sin\theta_r$ (ii)

iii) $\oint \bar{B} \cdot d\bar{a} = 0$

$$\Rightarrow B^{\perp}_{above} - B^{\perp}_{below} = 0$$

$$\Rightarrow \left(\hat{k}_I \wedge E_0 \cos\theta_i + \hat{k}_r \wedge E_0'' \cos\theta_i \right) = n \left(\hat{k}_T \wedge E_0' \cos\theta_r \right)$$

iv) $\oint \bar{H} \cdot d\bar{l} = \frac{d}{dt} \int \bar{D} \cdot d\bar{a} \Rightarrow$

$$\Rightarrow H^{\perp}_{above} - H^{\perp}_{below} = 0$$

$\Rightarrow$

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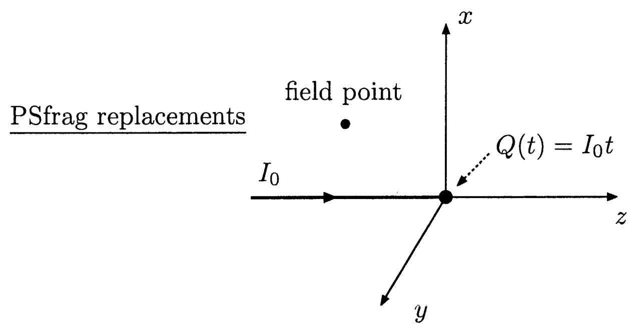
6. In the absence of polarizable and/or magnetizable material (i.e., only free charges and currents present) Maxwell's equations, in the Lorentz gauge, reduce to the inhomogeneous wave equation:

$$\Box \begin{Bmatrix} \Phi \\ A^x \\ A^y \\ A^z \end{Bmatrix} = \begin{Bmatrix} \rho/\epsilon_0 \\ \mu_0 J^x \\ \mu_0 J^y \\ \mu_0 J^z \end{Bmatrix}_{SI} = \frac{4\pi}{c} \begin{Bmatrix} c\rho \\ J^x \\ J^y \\ J^z \end{Bmatrix}_{Gaussian}, \text{ where } \Box \equiv \left(\frac{\partial}{c\partial t}\right)^2 - \nabla^2.$$



A time dependent charge $Q(t) = I_0\ t$, with $t \geq 0$ is fixed at the origin of a cylindrical coordinate system (r, ϕ, z). The charge increases with time, because of a constant current I_0 flowing along a thin wire attached to the charge from the left, see figure. Assume the wire carries no current for $t < 0$, however, at $t = 0$ a constant current I_0 abruptly flows in the $+z$ direction and remains constant for $t \geq 0$. Assume the wire remains neutral as the charge at the origin grows. Find the following quantities at time t for points (r, ϕ, z) where $z \leq -\sqrt{(ct)^2 - r^2}$ (this restriction on z and r simplifies the limits of integration of the integral you must evaluate for the vector potential).

Hint: The retarded Green's function for the $\Box$ operator is:

$$G^{ret}(\mathbf{r}, t; \mathbf{r}', t') = \frac{\delta\left(t - t' - |\mathbf{r} - \mathbf{r}'|/c\right)}{|\mathbf{r} - \mathbf{r}'|}.$$

(a) [2 points] The scalar potential Φ.

(b) [3 points] The vector potential $\mathbf{A}$.

(c) [2 points] The magnetic induction $\mathbf{B}$.

(d) [3 points] The electric field strength $\mathbf{E}$.