Quantum Mechanics
Qualifying Exam–January 2010

Notes and Instructions:

- There are **6** problems and **7** pages.
- Be sure to write your alias at the top of every page.
- Number each page with the problem number, and page number of your solution (e.g. "Problem 3, p. 1/4" is the first page of a four page solution to problem 3).
- **You must show all your work.**

Possibly useful formulas:

Pauli spin matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

One-dimensional simple harmonic oscillator operators:

$$X = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger)$$

$$P = -i\sqrt{\frac{\hbar m\omega}{2}}(a - a^\dagger)$$

Spherical Harmonics:

$$Y_0^0(\theta,\varphi) = \frac{1}{\sqrt{4\pi}} \qquad Y_2^2(\theta,\varphi) = \frac{5}{\sqrt{96\pi}}\, 3\sin^2\theta\, e^{2i\varphi}$$

$$Y_2^1(\theta,\varphi) = -\frac{5}{\sqrt{24\pi}}\, 3\sin\theta\cos\theta\, e^{i\varphi}$$

$$Y_1^1(\theta,\varphi) = -\frac{3}{\sqrt{8\pi}}\sin\theta\, e^{i\varphi} \qquad Y_2^0(\theta,\varphi) = \frac{5}{\sqrt{4\pi}}\left(\tfrac{3}{2}\cos^2\theta - \tfrac{1}{2}\right)$$

$$Y_1^0(\theta,\varphi) = \frac{3}{\sqrt{4\pi}}\cos\theta \qquad Y_2^{-1}(\theta,\varphi) = \frac{5}{\sqrt{24\pi}}\, 3\sin\theta\cos\theta\, e^{-i\varphi}$$

$$Y_1^{-1}(\theta,\varphi) = \frac{3}{\sqrt{8\pi}}\sin\theta\, e^{-i\varphi} \qquad Y_2^{-2}(\theta,\varphi) = \frac{5}{\sqrt{96\pi}}\, 3\sin^2\theta\, e^{-2i\varphi}$$

Angular momentum raising and lowering operators:

$$\hat{L}_\pm = (\hat{L}_x \pm i\,\hat{L}_y)$$

PROBLEM 1: The Delta-Function Potential

Let us consider a single particle of mass m moving in one dimension with the Hamiltonian

$$H = T + V(x) ,$$

where the kinetic energy is

$$T = \frac{P^2}{2m} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} ,$$

the potential energy is

$$V(x) = -V_0\, \delta(x) ,$$

and $\delta(x)$ is the Dirac delta function.

(a) [2 points] Find an expression for the discontinuity of the derivative of the wave function at $x = 0$.

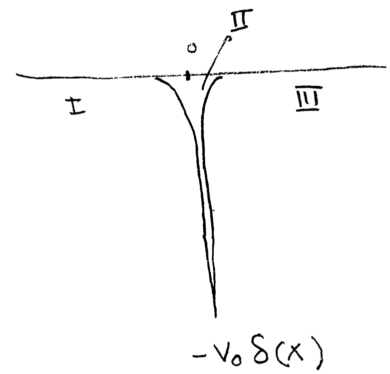
(b) [3 points] Find the ground state wave function.

(c) [2 points] Find the ground state energy.

(d) [3 points] Find the expectation value for the kinetic energy, $\langle T \rangle$.

Problem 1.

Lets consider $E<0$



- Region 1: $(-\infty < x < 0)$

$$d^2\psi_I(x) + \frac{2mE}{\hbar^2}\psi_I(x) = 0 \qquad \beta^2 = -\frac{2mE}{\hbar^2}$$

$$\Rightarrow \frac{d^2\psi_I(x)}{dx^2} = \beta^2\psi_I(x)$$

$$\Rightarrow \psi_I(x) = Ae^{\beta x} + Be^{-\beta x}$$

but $x \to -\infty$ $e^{-\beta x} \to \infty$ $\therefore B = 0$

$$\Rightarrow \psi_I(x) = Ae^{\beta x} \qquad (1)$$

- Region 3 $(\infty > x > 0)$

$$\frac{d^2\psi_{III}(x)}{dx^2} + \beta^2\psi_{III}(x) = 0$$

$$\Rightarrow \psi_{III}(x) = Ce^{-\beta x} + De^{\beta x}$$ but $x \to \infty$ $e^{\beta x} \to \infty$ $\therefore D = 0$

$$\Rightarrow \psi_{III}(x) = Ce^{-\beta x} \qquad (2)$$

* Even with a delta func the continuity of the wave func is required.

$$\psi_I(0) = \psi_{III}(0) \Rightarrow A = C$$

* The derivative of the wave func^c is not continuous. We can find the discontin by integrating the term $\frac{2m}{\hbar^2} V\delta(x)$

$$\frac{d^2\psi(x)}{dx^2} + \left[\frac{2m}{\hbar^2} V_0\, \delta(x) + \frac{2mE}{\hbar^2}\right]\psi_{II}(x) = 0$$

$$\Rightarrow \frac{d^2\psi(x)}{dx^2} + \left[\frac{2m}{\hbar^2} V_0\, \delta(x) - \beta^2\right]\psi(x) = 0$$

$$\Rightarrow \underset{\epsilon\to 0}{Lt} \int_{-\epsilon}^{+\epsilon} \frac{d^2\psi(x)}{dx^2}\, dx + \frac{2m}{\hbar^2}\int_{-\epsilon}^{+\epsilon} V_0\, \delta(x)\psi(x)\, dx - \underset{\epsilon\to 0}{Lt}\, \beta^2 \int_{-\epsilon}^{+\epsilon} \psi_{II}(x)\, dx$$

$$\Rightarrow \left.\frac{d\psi(x)}{dx}\right|_{x=+\epsilon} - \left.\frac{d\psi(x)}{dx}\right|_{x=-\epsilon} + \frac{2mV_0\psi(0)}{\hbar} = 0$$

$$\Rightarrow \Delta\left(\frac{d\psi(x)}{dx}\right) = -\frac{2mV_0}{\hbar^2}\psi(0)$$

$x = +\epsilon$ corrosponds to the wave fun^c in Region III

Using $\psi_{III}(x) = Ae^{-\beta x}$

$x = -\epsilon$ corrosponds to the wave fun^c in Region I

using $\psi_I(x) = Ae^{\beta x}$

$$\Delta\left(\frac{d\psi(x)}{dx}\right) = \psi'_{III}(x) - \psi'_I(x) = -\beta A e^{-\beta x}\Big|_{x=+\epsilon} - \beta A e^{\beta x}\Big|_{x=-\epsilon} = -\beta A\left(e^{-\beta\epsilon} + e^{\beta\epsilon}\right)$$

$$\underset{\epsilon\to 0}{Lt}\ \Delta\left(\frac{d\psi(x)}{dx}\right) = -2\beta A$$

↑ so this is the discontinuity

Further working.

Thus,

$$-2\beta A = -\frac{2mV_0}{\hbar^2} A \quad \Big| \text{ where } \psi(0) = A = C$$

$$\Rightarrow \beta = \frac{mV_0}{\hbar^2}$$

Since, $\beta^2 = \frac{-2mE}{\hbar^2} \Rightarrow E = -\frac{\beta^2\hbar^2}{2m} = -\frac{mV_0^2}{2\hbar^2}$

b) The wave func are

$$\left.\begin{aligned} \psi_I(x) &= A e^{+\beta x} \\ \psi_{III}(x) &= A e^{-\beta x} \end{aligned}\right\} \psi(x) = A e^{-\beta|x|}$$

$\Rightarrow \psi(x) = \sqrt{\beta}\, e^{-\beta|x|}$ it is a bound state, so this has to be the ground state.

c) $$E = -\frac{mV_0^2}{2\hbar^2}$$

d) $$\langle T \rangle = \int_{-\infty}^{+\infty} \psi^* \frac{\hat{p}^2}{2m} \psi\, dx = \frac{-\hbar^2\beta}{2m}\int_{-\infty}^{+\infty} e^{-\beta|x|} \frac{d^2}{dx^2} e^{-\beta|x|}\, dx$$

cannot do that → the derivative is not continuous from $-\infty \to +\infty$

$$= \frac{-\hbar^2\beta}{2m}\int_{-\infty}^{0} e^{+\beta x}\frac{d^2}{dx^2}e^{+\beta x}\, dx + \frac{-\hbar^2\beta}{2m}\int_{0}^{\infty} e^{-\beta x}\frac{d^2}{dx^2}e^{-\beta x}\, dx$$

$$= \frac{-\hbar^2\beta}{2m}\int_{-\infty}^{0} e^{+\beta x}\beta^2 e^{\beta x}\, dx + \frac{-\hbar^2\beta}{2m}\int_{0}^{\infty} e^{-\beta x}\beta^2 e^{-\beta x}\, dx$$

$$= \frac{-\hbar^2\beta^3}{2m}\left[\int_{-\infty}^{0} e^{+2\beta x}\,dx + \int_{0}^{\infty} e^{-2\beta x}\,dx\right]$$

$$= \frac{-\hbar^2\beta^3}{2m}\left[\frac{1}{2\beta} e^{2\beta x}\Big|_{-\infty}^{0} - \frac{1}{2\beta} e^{-2\beta x}\Big|_{0}^{\infty}\right]$$

$$= \frac{-\hbar^2\beta^2}{4m}\left[1+1\right] = \frac{-\hbar^2\beta^2}{2m} = \frac{V_0^2}{2}. = \langle T\rangle$$

Find $\langle V\rangle$ & $\langle E\rangle$ then $\langle T\rangle = \langle E\rangle - \langle V\rangle$

$$\langle V\rangle = \int_{-\infty}^{+\infty} \psi^* V \psi\,dx = \beta\int_{-\infty}^{+\infty} e^{-2\beta|x|}(-V_0\,\delta(x)) = -\beta V_0 \int_{-\infty}^{+\infty} e^{-2\beta|x|}\,\delta(x)\,dx$$

$$= -\beta V_0\left[\cancel{\int_{-\infty}^{0} e^{2\beta x}\,\delta(x)\,dx + \int_{0}^{\infty} e^{-2\beta|x|}}\,\delta(x)\,dx\right] = -\beta V_0 = -\frac{mV_0^2}{\hbar^2}$$

$$= \cancel{-\beta V_0\left[\frac{1}{2}\right.}$$

$$\langle E\rangle = \beta\int_{-\infty}^{+\infty} e^{-2\beta|x|}\left(-\frac{mV_0^2}{2\hbar^2}\right)dx = -\beta\frac{mV_0^2}{2\hbar^2}\left[\int_{-\infty}^{0} e^{+2\beta x}\,dx + \int_{0}^{\infty} e^{-2\beta x}\,dx\right]$$

$$= -\beta\frac{mV_0^2}{2\hbar^2}\left[\frac{1}{2\beta} e^{2\beta x}\Big|_{-\infty}^{0} - \frac{1}{2\beta} e^{-2\beta x}\Big|_{0}^{\infty}\right] = -\frac{mV_0^2}{2\hbar^2}$$

* Since E is const we can directly say $\langle E\rangle = -\frac{mV_0^2}{2\hbar^2}$

$$\langle T\rangle = \langle E\rangle - \langle V\rangle = -\frac{mV_0^2}{\hbar^2} + \frac{mV_0^2}{2\hbar^2} = \frac{mV_0^2}{2\hbar^2}$$

To calculate $\langle T \rangle$ directly

$$\langle T \rangle = \beta \int_{-\infty}^{+\infty} \psi^*(x) \frac{p^2}{2m} \psi(x)\,dx = -\frac{\hbar^2}{2m}\beta \int_{-\infty}^{+\infty} e^{-\beta|x|} \frac{d^2}{dx^2} e^{-\beta|x|}\,dx$$

$$= -\frac{\hbar^2}{2m}\beta \left[\int_{-\infty}^{-\epsilon} e^{+\beta x} \frac{d^2}{dx^2} e^{\beta x}\,dx \right] + \int_{-\epsilon}^{+\epsilon} \psi^*(x)\, T\, \psi(x)\,dx - \frac{\hbar^2 \beta}{2m}\left[\int_{+\epsilon}^{\infty} e^{-\beta x} \frac{d^2}{dx^2} e^{-\beta x} \right]$$

$$\Rightarrow \int_{-\infty}^{-\epsilon} e^{+\beta x} \frac{d^2}{dx^2} e^{\beta x}\,dx = \int_{-\infty}^{-\epsilon} e^{\beta x} \beta^2 e^{\beta x}\,dx = \beta^2 \int_{-\infty}^{-\epsilon} e^{2\beta x}\,dx = \frac{\beta}{2} e^{2\beta x}\Big|_{-\infty}^{-\epsilon}$$

$$= \frac{\beta}{2} e^{-2\beta\epsilon}$$

As, $\epsilon \to 0$ $\quad = \dfrac{\beta}{2}$

$$\Rightarrow \int_{+\epsilon}^{\infty} e^{-\beta x} (\beta^2) e^{-\beta x}\,dx = \beta^2 \int_{+\epsilon}^{\infty} e^{-2\beta x}\,dx = -\frac{\beta}{2} e^{-2\beta x}\Big|_{+\epsilon}^{\infty} = +\frac{\beta}{2} e^{-2\beta\epsilon}$$

$\epsilon \to 0$ $\quad = \dfrac{\beta}{2}$

$$\Rightarrow \int_{-\epsilon}^{+\epsilon} \psi^*(x)\, T\, \psi(x)\,dx = \int_{-\epsilon}^{+\epsilon} \psi^*(x)\,(H-V)\,\psi(x)\,dx = \int_{-\epsilon}^{+\epsilon} \psi^*(x)\,(E-V)\,\psi(x)\,dx$$

$$= \int_{-\epsilon}^{+\epsilon} \psi^*_{(x)} E \psi(x)\,dx - \int_{-\epsilon}^{+\epsilon} \psi^*(x)\,(-V_0\delta(x))\,\psi(x)$$

$$= E \underbrace{\int_{-\epsilon}^{+\epsilon} \psi^*(x)\psi(x)\,dx}_{\to 0} + V_0 \underbrace{|\psi(0)|^2}_{\beta}$$

As $\epsilon \to 0$ $\quad = +\beta V_0$

$$\langle T \rangle = -\frac{\hbar^2\beta^2}{4m} - \beta V_0 - \frac{\hbar^2\beta^2}{4m} = -\frac{\hbar^2}{4m} \times \frac{m^2 V_0^2}{\hbar^4} + \frac{mV_0^2}{\hbar^2} = \frac{mV_0^2}{2\hbar^2}$$

5-10

PROBLEM 2: Hydrogenic Atoms with One Electron

In terms of the first Bohr radius, $a_0 \equiv \hbar/(c\alpha m_e)$, where α is the fine-structure constant, the ground-state eigenfunction of a hydrogen atom is

$$\psi_{1,0,0}(r,\theta,\varphi) = \frac{1}{\sqrt{\pi a_0^3}}\, e^{-r/a_0}.$$

(a) [3 points] Evaluate the probability of finding an electron in the ground-state of a hydrogen atom in the classically forbidden region. The classically forbidden region is the region of space where the classical kinetic energy is negative.

(b) [4 points] For the ground state, evaluate the uncertainty in the Cartesian coordinate x and the uncertainty in the corresponding component of the linear momentum, p_x. *Hint: You need not use the explicit form of the operator for the linear momentum to evaluate Δp_x.*

(c) [3 points] Show explicitly that the product of your uncertainties, $\Delta x\, \Delta p_x$, is consistent with the Heisenberg uncertainty principle.

didn't finish

$\langle p_x \rangle$ & $\langle p_x^2 \rangle$

$$\psi_{100}(r,\theta,\phi) = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0}$$

a) the classically allowed region $0 \leq r \leq a_0$ and beyond that the energy is negative which is the classically forbidden.

So, the probability

$$P(0 \leq r \leq a) = \int_0^{a_0} \psi_{100}^*(r,\theta,\phi)\psi_{100}(r,\theta,\phi)\, r^2 dr \sin\theta d\theta d\phi$$

$$= 4\pi \int_0^r \frac{1}{\pi a_0^3} e^{-2r/a_0} r^2 dr$$

$$= \frac{4}{a_0^3} \int_0^r r^2 e^{-2r/a_0} dr$$

$$= \frac{4}{a_0^3} \left\{ \frac{-a_0 e^{-2r/a_0}}{2} \left(r^2 - \frac{2r}{-2/a_0} + \frac{2}{(-2/a_0)^2} \right) \right\}_0^{a_0}$$

$$= \frac{4}{a_0^3} \left\{ \frac{-a_0 e^{-2r/a_0}}{2} \left(r^2 + r a_0 + \frac{a_0^2}{2} \right) \right\}_0^{a_0}$$

$$= \frac{4}{a_0^3} \left\{ -\frac{a_0 e^{-2}}{2} \left(a_0^2 + a_0^2 + \frac{a_0^2}{2} \right) \right\} + 1$$

$$= \frac{4}{a_0^3} \left\{ -\frac{5a_0^3}{4} e^{-2} \right\} + 1 = -5e^{-2} + 1 = 1 - 5e^{-2}$$

$$P(a_0 \leq r < \infty) = 1 - 1 + 5e^{-2} = 5e^{-2}$$

b. $\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$

$x = r \sin\theta \cos\phi$

$\langle x\rangle \int \psi^*_{00}(r,\theta,\phi)\ r \sin\theta \cos\phi\ \psi_{000}(r,\theta,\phi)$

$= \int \frac{1}{\pi a_0^3} e^{-r/a_0}\ r \sin\theta\cos\phi\ e^{-r/a_0}\ r^2\, dr \sin\theta\, d\theta\, d\phi$

$= \frac{1}{\pi a_0^3} \int_{0,0,0}^{r,\pi,2\pi} r^3 e^{-2r/a_0} \sin^2\theta \cos\phi\ dr\, d\theta\, d\phi$

$= \frac{2\pi}{\pi a_0^3} \int_0^r r^3 e^{-2r/a_0}\, dr \int_{\theta=0}^{\pi} \sin^2\theta\, d\theta \underbrace{\int_{\phi=0}^{2\pi} \cos\phi\, d\phi}_{\sin\phi\,\Big|_0^{2\pi}} = 0$

$= 0$

$\langle x^2\rangle = \underbrace{\frac{1}{\pi a_0^3}\int_0^r r^4 e^{-2r/a_0}\, dr}\ \underbrace{\int_{\theta=0}^{\pi} \sin^3\theta\, d\theta}\ \underbrace{\int_{\phi=0}^{2\pi} \cos^2\phi\, d\phi}_{\pi}$

$\left[\cos\cancel{\theta} + \frac{1}{3}\cos^3\theta\right]_0^\pi$

$\left[1 - \frac{1}{3} + 1 - \frac{1}{3}\right]$

$= \frac{4}{3}$

$$\frac{1}{\pi a_0^3}\int_0^\infty r^4 \, e^{-2r/a_0}\, dr$$

$$= \frac{1}{\pi a_0^3}\left[\frac{-a_0 e^{-2r/a_0}}{2}\left\{r^4 - \frac{4r^3}{-2/a_0} + \frac{4\cdot 3\, r^2}{(-2/a_0)^2} - \frac{4\cdot 3\cdot 2 r}{(-2/a_0)^3} + \frac{4!}{(-2/a_0)^4}\right\}\right]_0^\infty$$

$$= \frac{1}{\pi a_0^3}\left[-\frac{a_0 e^{-r/a_0}}{2}\left\{r^4 + 2r^3 a_0 + 3r^2 a_0^2 + 3 r a_0^3 + \frac{3}{2}a_0^4\right\}\right]_0^\infty$$

$$= \frac{3\, a_0^2}{\pi 4}$$

$$\langle x^2 \rangle = a_0^2$$

$$\Delta x = a_0$$

b) $$\psi_{100}(x,y,z) = \frac{1}{\sqrt{\pi a_0^3}}\, e^{-(x^2+y^2+z^2)^{1/2}/a_0}$$

$$\langle p_x \rangle = \int \psi_{100}^* \, p_x \, \psi_{100} \, dx\,dy\,dz$$

$$= \frac{1}{\pi a_0^3}\int e^{-\frac{(x^2+y^2+z^2)^{1/2}}{a_0}} (-i\hbar)\frac{d}{dx}\, e^{-\frac{(x^2+y^2+z^2)^{1/2}}{a_0}} \, dx\,dy\,dz$$

$$= \frac{-i\hbar}{\pi a_0^3}\int e^{-\frac{(x^2+y^2+z^2)^{1/2}}{a_0}} \left(\frac{-1}{a_0}\right)\frac{1}{2}(x^2+y^2+z^2)^{-\frac{1}{2}}(2x)\, e^{-\frac{(x^2+y^2+z^2)^{1/2}}{a_0}}\, dx\,dy\,dz$$

$$= \frac{i\hbar}{\pi a_0^{\cancel{3}4}}\int \frac{x\, e^{-\frac{2(x^2+y^2+z^2)^{1/2}}{a_0}}}{(x^2+y^2+z^2)^{\frac{1}{2}}}\, dx\,dy\,dz = \frac{i\hbar}{\pi a_0^4}\int \frac{r\sin\theta\cos\phi \; e^{-\frac{2r}{a_0}}}{r}\, r^2 dr \sin\theta \, d\theta \, d\phi$$

$$\langle p_x \rangle = \frac{i\hbar}{a_0^4} \int_0^r r^2 e^{-2r/a_0} dr \int_{\theta=0}^{\pi} \sin^2\theta \, d\theta \int_{\phi=0}^{2\pi} \cos\phi \, d\phi = 0$$

$$\langle p_x^2 \rangle = \underbrace{\frac{-\hbar^2}{\pi a_0^3}}_{\alpha} \int e^{-\frac{(x^2+y^2+z^2)^{\frac{1}{2}}}{a_0}} \frac{d^2}{dx^2} e^{-\frac{(x^2+y^2+z^2)^{\frac{1}{2}}}{a_0}} \, dx\,dy\,dz$$

$$= \alpha \int e^{-r/a_0} \frac{d}{dx} \left\{ -\frac{1}{a_0} \frac{1}{2} r^{-\frac{1}{2}} (2x) e^{-\frac{(x^2+y^2+z^2)^{\frac{1}{2}}}{a_0}} \right\} dx\,dy\,dz$$

$$= \frac{\hbar^2}{\pi a_0^4} \int e^{-r/a_0} \frac{d}{dx} \left\{ x \,(x^2+y^2+z^2)^{-\frac{1}{2}} e^{-\frac{(x^2+y^2+z^2)^{\frac{1}{2}}}{a_0}} \right\} dx\,dy\,dz$$

$$= \alpha \int e^{-r/a_0} \left\{ r^{-\frac{1}{2}} e^{-r/a_0} + x\left(-\frac{1}{2}\right) r^{-\frac{3}{2}} (2x)\, e^{-r/a_0} + x\, r^{-\frac{1}{2}} \left(-\frac{1}{a_0}\right) \frac{1}{2} r^{-\frac{1}{2}} (2x)\, e^{-r/a_0} \right\} dx\,dy\,dz$$

$$= \alpha \int e^{-2r/a_0} r^{3/2} \, dr\, d\Omega - \underbrace{\int e^{-2r/a_0} x^2 r^{-3/2} r^2 \, dr\, d\Omega}_{-\int e^{-2r/a_0} r^2 \sin^2\theta \cos^2\phi \, r^{\frac{1}{2}} dr\, d\Omega - \frac{1}{a_0} \int r^2} - \frac{1}{a_0} \int x^2 r^{-1} e^{-2r/a_0} r^2 dr\, d\Omega$$

I thought TDPT not in syllabus?

PROBLEM 3: Time-Dependent Perturbation Theory

Consider a non-relativistic particle of mass m and charge q with the potential energy:

$$V(x) = \frac{1}{2}\,k\,X^2$$

A homogeneous electric field $\mathcal{E}(t)$ directed along the x-axis is switched on at time $t = 0$. This causes a perturbation of the form

$$H' = -q\,X\,\mathcal{E}(t)$$

where $\mathcal{E}(t)$ has the form

$$\mathcal{E}(t) = \mathcal{E}_o e^{-t/\tau}$$

where $\mathcal{E}_o$ and τ are constants.

The particle is in the ground state at time $t \leq 0$. This problem will deal with calculating the probability that it will be found in an excited state as $t \longrightarrow \infty$.

The probability that the particle makes a transition from an initial state i to a final state f is given by:

$$P_{fi}(t, t_o) = \frac{1}{\hbar^2}\left|\int_{t_o}^{t} dt'\,\langle\phi_f|H'(t')|\phi_i\rangle\,e^{i\omega_{fi}t'}\right|^2 .$$

where the particle originally is in state ϕ_i and finally in state ϕ_f.

(a) [2 points] In terms of known quantities, what is the value of ω_{fi} ?

(b) [2 points] How many excited states can the particle make a transition to?

(c) [6 points] Derive an expression for the probability that the particle will be found in any allowed excited state as $t \longrightarrow \infty$.

PROBLEM 4: Spin Physics

Spin-1/2 objects generally have magnetic moments that affect their energy levels and dynamics in magnetic fields. The interaction between the magnetic moment and a magnetic field, $\vec{B}$ can be written as:

$$H = -\mu \vec{S} \cdot \vec{B} \tag{1}$$

where $\vec{S}$ is the spin of the particle

$$\vec{S} = \frac{\hbar}{2}\vec{\sigma} \tag{2}$$

where the σ_i's are Pauli matrices.

In this problem we'll be using as our basis the eigenstates of S_z,

$$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3}$$

with eigenvalues $\pm\frac{\hbar}{2}$.

(a) [1 point] If a particle is in the spin state $|+\rangle$, compute the expectation values of S_x, S_y, and S_z.

(b) [1 point] If a particle is in the spin state $|+\rangle$, what are the uncertainties of S_x, S_y, and S_z? ($\Delta S_i^2 = \langle S_i^2\rangle - \langle S_i\rangle^2$.) Explain the physics of your results in terms of the eigenvalues and measurement probabilities of the spin in the x, y, and z directions. to IK

(c) [3 points] A large ensemble of particles are all prepared to be in the spin state $|+\rangle$ at time $t = 0$ when a magnetic field in the x-direction is switched on, $\vec{B} = B_0\hat{e}_x$. Solve for the time-dependent probabilities, $P_\pm(t)$, of measuring S_z to be $\pm\hbar/2$.

(d) [2 points] For the experiment described in part (c), what are the probabilities for measuring S_x to be $\pm\hbar/2$? Explain the differences between the results for S_z and S_x.

(e) [3 points] Consider the case where the magnetic field is $\vec{B} = \frac{B_0}{\sqrt{2}}(\hat{e}_x + \hat{e}_z)$. In this case what is the time-dependent probability of measuring S_z to be $+\hbar/2$?

P4

a) $S_x = \frac{\hbar}{2}\sigma_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow S_x^2 = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$S_y = \frac{\hbar}{2}\sigma_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \Rightarrow S_y^2 = \frac{\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$S_z = \frac{\hbar}{2}\sigma_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow S_z^2 = \frac{\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\langle \hat{S}_x \rangle = \langle + | \hat{S}_x | + \rangle = (1 \ 0)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{\hbar}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2}(1 \ 0)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$

$\langle \hat{S}_y \rangle = (1 \ 0)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\frac{\hbar}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2}(1 \ 0)\begin{pmatrix} 0 \\ i \end{pmatrix} = 0$

$\langle \hat{S}_z \rangle = \frac{\hbar}{2}(1 \ 0)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2}(1 \ 0)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2}$

b) $\langle S_x^2 \rangle = \frac{\hbar^2}{4}(1 \ 0)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar^2}{4}(1 \ 0)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar^2}{4}$

$\langle S_y^2 \rangle = \frac{\hbar^2}{4}$

$\langle S_z^2 \rangle = \frac{\hbar^2}{4}$

$\Delta S_x = \frac{\hbar}{2} \qquad \Delta S_y = \frac{\hbar}{2} \qquad \Delta S_z = 0$

Since, the state $|+\rangle$ is in S_z-basis we can measure the probability without any uncertainty but since the state is not in the eigenbasis of S_x or S_y we will have probability < 1 which indicates uncertainty.

$$H = -\mu \bar{S} \cdot \bar{B}$$

c) $$|\psi(0)\rangle = |+\rangle_z$$

$$= -\mu S_x B_0$$

$$|\psi(t)\rangle = U |\psi(0)\rangle$$

Now,

$$H = -\mu \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} B_0 = \begin{pmatrix} 0 & -\mu B_0 \\ -\mu B_0 & 0 \end{pmatrix} = \begin{vmatrix} -\lambda & -\mu B_0 \\ -\mu B_0 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 - (\mu B_0)^2 = 0 \Rightarrow \lambda = \pm \mu B_0$$

- $\underline{\lambda = +\mu B_0}$ $$\begin{pmatrix} -\mu B_0 & -\mu B_0 \\ -\mu B_0 & -\mu B_0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0 \Rightarrow a_2 = -a_1$$

$$\rightsquigarrow |E = +\mu B_0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \left[|+\rangle - |-\rangle \right]$$

- $\underline{\lambda = -\mu B_0}$ $$\begin{pmatrix} +\mu B_0 & -\mu B_0 \\ -\mu B_0 & +\mu B_0 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = 0 \Rightarrow b_2 = +b_1.$$

$$\rightsquigarrow |E = -\mu B_0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \left[|+\rangle + |-\rangle \right]$$

Now,

$$|+\rangle_E = \frac{1}{\sqrt{2}} \left[|E = +\mu B_0\rangle + |E = -\mu B_0\rangle \right]$$

$$|\psi(t)\rangle = \sum_E e^{-iEt/\hbar} |\psi(0)\rangle_E$$

$$= \frac{1}{\sqrt{2}} e^{-i\frac{\mu B_0 t}{\hbar}} |E = +\mu B_0\rangle + \frac{1}{\sqrt{2}} e^{+i\frac{\mu B_0 t}{\hbar}} |E = -\mu B_0\rangle$$

$$= \frac{1}{2} e^{-i\frac{\mu B_0 t}{\hbar}} \left[|+\rangle_z - |-\rangle_z\right] + \frac{1}{2} e^{+i\frac{\mu B_0 t}{\hbar}} \left[|+\rangle_z + |-\rangle_z\right]$$

$$P(S_z = +\tfrac{\hbar}{2}) = \frac{|\langle + | \psi(t)\rangle|^2}{\langle\psi(t)|\psi(t)\rangle}$$

$$= \left| \frac{1}{2} e^{-i\mu B_0 t/\hbar} + \frac{1}{2} e^{i\mu B_0 t/\hbar} \right|^2$$

$$= \left| \cos(\frac{\mu B_0}{\hbar}) t \right|^2 = \cos^2(\frac{\mu B_0}{\hbar}) t$$

$$P(S_z = -\hbar/2) = \frac{|\langle - | \psi(t)\rangle|^2}{\langle\psi(t)|\psi(t)\rangle} = \left| -\frac{1}{2} e^{-i\mu B_0 t/\hbar} + \frac{1}{2} e^{i\mu B_0 t/\hbar} \right|$$

$$= \left| i \sin(\frac{\mu_0 B}{\hbar}) t \right|^2 = \sin^2\left(\frac{\mu_0 B}{\hbar}\right) t$$

c) $S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow \begin{vmatrix} -\lambda & \frac{\hbar}{2} \\ \frac{\hbar}{2} & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - \frac{\hbar^2}{4} = 0$

$\Rightarrow \lambda = \pm \frac{\hbar}{2}$

- $\underline{\lambda = \frac{\hbar}{2}}$

$$\begin{pmatrix} -\frac{\hbar}{2} & \frac{\hbar}{2} \\ \hbar/2 & -\hbar/2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0 \Rightarrow a_1 = a_2$$

$$|\lambda = +\tfrac{\hbar}{2}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow |+\rangle_x = \frac{1}{\sqrt{2}}\left[|+\rangle_z + |-\rangle_z\right]$$

- $\underline{\lambda = -\frac{\hbar}{2}}$

$$\begin{pmatrix} \hbar/2 & \hbar/2 \\ \hbar/2 & \hbar/2 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = 0 \Rightarrow b_2 = -b_1 \quad |\lambda = -\tfrac{\hbar}{2}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\Rightarrow |-\rangle_x = \frac{1}{\sqrt{2}}\left[|+\rangle_z - |-\rangle_z\right]$$

$$P\left(S_x = \tfrac{\hbar}{2}\right) = \frac{|{}_x\langle + | \psi(t)\rangle|^2}{\langle \psi(t)|\psi(t)\rangle}$$

$$= \left| \frac{1}{2\sqrt{2}} e^{-i\frac{\mu B_0}{\hbar}t} - \frac{1}{2\sqrt{2}} e^{-i\frac{\mu B_0 t}{\hbar}} + \frac{1}{2\sqrt{2}} e^{i\frac{\mu B_0}{\hbar}t} + \frac{1}{2\sqrt{2}} e^{i\frac{\mu B_0 t}{\hbar}} \right|$$

$$= \left| \frac{1}{\sqrt{2}} e^{i2\frac{\mu B_0}{\hbar}t} \right|^2$$

$$= \frac{1}{2}$$

Note: Time independent as $H = B_0 S_x$

S_x is the eigenbasis of H

$$P\left(S_x = -\tfrac{\hbar}{2}\right) = -\frac{1}{2}$$

e) $\bar{B} = \frac{B_0}{\sqrt{2}}(\hat{e}_x + \hat{e}_z)$

$$H = -\mu\, \vec{S}\cdot\vec{B} = -\frac{\mu}{\sqrt{2}}(S_x B_0 + S_y B_0)$$

PROBLEM 5: Two Level System

Consider a quantum system that can be accurately approximated as having two energy levels $|+\rangle$ and $|-\rangle$ such that

$$H_0|\pm\rangle = \pm\epsilon|\pm\rangle\,,$$

where ϵ is energy.

When placed in an external field, the eigenstates of H_0 are mixed by another term in the total Hamiltonian

$$V|\pm\rangle = \delta|\mp\rangle\,.$$

For simplicity, we choose ϵ to be real.

(a) [1 points] Using the states $|+\rangle$ and $|-\rangle$ as your basis states, write down the matrix representations for the operators H_0 and V.

(b) [3 points] What will be the possible results if a measurement is made of the energy for the full Hamiltonian $H = H_0 + V$?

(c) [2 points] Experiments are performed that measure the transition energies between eigenstates. Without the external field ($\delta = 0$) it is found that the transition energy is 4 eV and with the external field ($\delta \neq 0$) the transition energy is 6 eV. What is the coupling between the states $|\pm\rangle$, δ, in this case?

(d) [2 points] We can write the eigenstates of the total Hamiltonian in terms of two energy levels $|\pm\rangle$ as

$$\begin{aligned}|1\rangle &= \cos(\theta_1)|+\rangle + \sin(\theta_1)|-\rangle \\ |2\rangle &= \cos(\theta_2)|+\rangle + \sin(\theta_2)|-\rangle\,.\end{aligned}$$

Letting $\delta/\epsilon = C$, solve for the angles θ_1 and θ_2 in terms of C.

(e) [2 points] Consider an experiment where the two-level system starts in the eigenstate of H_0 with eigenvalue $-\epsilon$. A very weak field is turned on so that $C \ll 1$. To the lowest order in C, what is the probability of measuring a positive energy for the system when $\delta \neq 0$?

P5

$$H_0|+\rangle = +\epsilon|+\rangle$$

$$H_0|-\rangle = -\epsilon|-\rangle$$

$$V|+\rangle = \delta|-\rangle$$

$$V|-\rangle = \delta|+\rangle$$

a)

$$H_0 = \begin{array}{c|cc} & |+\rangle & |-\rangle \\ \hline \langle +| & \epsilon & 0 \\ \langle -| & 0 & -\epsilon \end{array}$$

$\langle +|H|+\rangle = \epsilon$

$\langle +|H|-\rangle = = 0$

$\langle -|H|-\rangle = -\epsilon$

$\langle -|H|+\rangle = 0$

$$V = \begin{array}{c|cc} & |+\rangle & |-\rangle \\ \hline \langle +| & 0 & \delta \\ \langle -| & \delta & 0 \end{array}$$

$\langle +|V\cancel{H}|+\rangle = 0$

$\langle +|V|-\rangle = \delta$

$\langle -|V|+\rangle = \delta$

$\langle -|V|-\rangle = 0$

b)

$$H = H_0 + V = \begin{pmatrix} \epsilon & \delta \\ \delta & -\epsilon \end{pmatrix}$$

$$\begin{vmatrix} \epsilon-\lambda & \delta \\ \delta & -\epsilon-\lambda \end{vmatrix} = 0$$

$$\Rightarrow -(\epsilon-\lambda)(\epsilon+\lambda) - \delta^2 = 0$$

$$\Rightarrow -\epsilon^2 + \lambda^2 - \delta^2 = 0$$

$$\Rightarrow \lambda^2 - (\epsilon^2 + \delta^2) = 0$$

$$\Rightarrow \lambda = \pm\sqrt{\epsilon^2+\delta^2}$$

c) $\langle + | H_0 | - \rangle = 4eV$

$\langle + | H_0 + V | - \rangle = 6ev$

$\Rightarrow \langle + | H_0 | - \rangle + \langle + | V | - \rangle = 64eV$

$\Rightarrow \delta = 2eV$

d) $H|1\rangle = H\left(\cos(\theta_1)|+\rangle + \sin\theta_1 |-\rangle\right)$

$= \cos\theta_1 (H_0+V)|+\rangle + \sin\theta_1 (H_0+V)|-\rangle$

$= \cos\theta_1 \epsilon |+\rangle + \cos\theta_1 \delta |-\rangle - \sin\theta_1 \epsilon |-\rangle$
$+ \sin\theta_1 \delta |+\rangle$

$= \epsilon\left(\cos\theta_1 + \frac{\delta}{\epsilon}\sin\theta_1\right)|+\rangle + \epsilon\left(\frac{\delta}{\epsilon}\cos\theta_1 - \sin\theta_1\right)|-\rangle$

$= \epsilon\left[\cos\theta_1 + C\sin\theta_1\right]|+\rangle - \epsilon\left[\sin\theta_1 - C\cos\theta_1\right]|-\rangle$

$H|2\rangle = (H_0+V)\left[\cos\theta_2|+\rangle + \sin\theta_2|-\rangle\right]$

$= \epsilon\left[\cos\theta_2 + C\sin\theta_2\right]|+\rangle - \epsilon\left[\sin\theta_2 - C\cos\theta_2\right]|-\rangle$

Now,

$\epsilon\left[\cos\theta_1 + C\sin\theta_1\right] = \sqrt{\epsilon^2+\delta^2}\cos\theta_1 \quad (1)$

$\epsilon\left[C\cos\theta_1 - \sin\theta_1\right] = +\sqrt{\epsilon^2+\delta^2}\sin\theta_1 \quad (2)$

$$\cos\theta_1(1+c) + \cancel{\sin\theta_1}(c-1) = \frac{\sqrt{\epsilon^2+\delta^2}}{\epsilon}(\cos\theta_1 + \sin\theta_1)$$

$$\Rightarrow (\cos\theta_1 - \sin\cancel{\theta_1}) + c(\cos\theta_1 + \sin\theta_1) = \frac{\sqrt{\epsilon^2+\delta^2}}{\epsilon}(\cos\theta_1 + \sin\theta_1)$$

(1) $\Rightarrow$

$$\cos\theta_1 + c\sin\theta_1 = \sqrt{1+c^2}\cos\theta_1$$

$$\Rightarrow \quad c\sin\theta_1 = \left(\sqrt{1+c^2} - 1\right)\cos\theta_1$$

$$\Rightarrow \quad \theta_1 = \tan^{-1}\frac{\sqrt{1+c^2}-1}{c}$$

(2) $\Rightarrow$

$$c\cos\theta_1 - \sin\theta_1 = \sqrt{1+c^2}\sin\theta_1$$

$$\Rightarrow \quad c\cos\theta_1 = \left(\sqrt{1+c^2}+1\right)\sin\theta_1$$

$$\Rightarrow \quad \tan\theta_1 = \frac{c}{\sqrt{1+c^2}+1}$$

$$= \frac{c\left(\sqrt{1+c^2}-1\right)}{1+c^2-1} = \frac{\sqrt{1+c^2}-1}{c}$$

$$\Rightarrow \theta_1 = \tan^{-1}\frac{\sqrt{1+c^2}-1}{c}$$

checks out

Similarly we can do θ_2

$$\epsilon\left[\cos\theta_2 + c\sin\theta_2\right] = -\sqrt{\epsilon^2+\delta^2}\cos\theta_2$$

$$\Rightarrow \cos\theta_2 + c\sin\theta_2 = -\sqrt{1+c^2}\cos\theta_2$$

$$\Rightarrow \left(1+\sqrt{1+c^2}\right)\cos\theta_2 = -c\sin\theta_2$$

$$\Rightarrow \tan\theta_2 = -\frac{1+\sqrt{1+c^2}}{c}$$

$$\Rightarrow \theta_2 = \tan^{-1}\left\{-\frac{1+\sqrt{1+c^2}}{c}\right\}$$

e) $$P(|1\rangle) = \frac{|\langle 1|-\rangle|^2}{\langle -|-\rangle}$$

 5-10

PROBLEM 6: Hyperfine Splitting

The hyperfine splitting in hydrogen comes from a spin-spin interaction between the electron and the proton. The total Hamiltonian can be written as

$$H = \frac{P_p^2}{2m_p} + \frac{P_e^2}{2m_e} - \frac{e^2}{r} + H_{HF}$$

where $H_{HF} = A\vec{S}_e \cdot \vec{S}_p$, and A is a real constant.

(a) [1 points] What are the spin quantum numbers s and m_s of the electron?

(b) [1 points] What are the spin quantum numbers s and m_s of the proton?

(c) [1 points] What are the spin quantum numbers s and m_s of the combined electron-proton system?

Cannot do it

(d) [5 points] Diagonalize H_{HF} in the total $\vec{S} = \vec{S}_e + \vec{S}_p$ basis and compute the energy eigenvalues.

(e) [2 points] Write an expression for the energy of a photon that would be emitted from a hyperfine transition in terms of A, $\hbar$, and any other relevant constants.

??

P6.

a) $s_e = \frac{1}{2} \qquad m_{s_e} = \pm\frac{1}{2}$

b) $s_p = \frac{1}{2} \qquad m_{s_p} = \pm\frac{1}{2}$

c) $\vec{S} = |s_e + s_p| \; . \; . \; . \; |s_e - s_p|$

$= 0, 1$

$S = 0, \quad m_s = 0$

$S = 1, \quad m_s = 1, 0, -1$

d) $H_{HF} = A \, \vec{S}_e \cdot \vec{S}_p = A \left(S_{ex} S_{px} + S_{ey} S_{py} + S_{ez} S_{pz} \right)$

$= A \left(S_x^2 + S_y^2 + S_z^2 \right)$

$S_x = \frac{\hbar}{2} \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\hbar}{2}$

$S_y = \frac{\hbar}{2} \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \frac{\hbar}{2} \qquad H_{HF} = \frac{A\hbar^2}{4} \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$

$S_z = \frac{\hbar}{2} \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\hbar}{2}$

$S_x^2 = \frac{\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad S_y^2 = \frac{\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$S_z^2 = \frac{\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$|s=0, m_s=0\rangle$

$|s=1, m_s=1\rangle$ $|s=1, m_s=0\rangle$ $|s=1, m_s=-1\rangle$

$$\vec{S}_e = \begin{pmatrix} 1 & 1-i \\ 1+i & -1 \end{pmatrix} \frac{\hbar}{2} \qquad \vec{S}_p = \begin{pmatrix} 1 & 1-i \\ 1+i & -1 \end{pmatrix} \frac{\hbar}{2}$$

$$\vec{S} = \vec{S}_e + \vec{S}_p = \hbar \begin{pmatrix} 1 & 1-i \\ 1+i & -1 \end{pmatrix}$$

$$\begin{vmatrix} 1-\lambda & 1-i \\ 1+i & -1-\lambda \end{vmatrix} = 0 \quad \Rightarrow -(1-\lambda)(1+\lambda) - (1-i)(1+i) = 0$$

$$\Rightarrow -(1-\lambda)(1+\lambda) - (1-i^2) = 0$$

$$\Rightarrow -(1-\lambda^2) - 2 = 0$$

$$\Rightarrow -1 + \lambda^2 - 2 = 0$$

$$\Rightarrow$$