Problem 1: Spin $\frac{1}{2}$ particles (10 points)

Consider a system made up of spin 1/2 particles. If one measures the spin of the particles, one can only measure spin up or spin down. The general spin state of a spin 1/2 particle can be expressed as a two-element column matrix.

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix}$$

The spin matrices are:

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

a) Can one simultaneously measure S_x, S_y and S_z? Explain your answer. (1 pt)

b) Can one simultaneously measure S^2 and S_z? Explain your answer. (1 pt)

c) Show S_z is Hermetian. (1 pt)

d) Calculate the normalized eigenvectors and eigenvalues of S_z. (2 pts)

Suppose a spin 1/2 particle is in the state

$$\chi = A\begin{pmatrix} 1+i \\ 2 \end{pmatrix}$$

e) Normalize the state in order to determine A (1 pt)

f) If one measures S_z, what is the probability of getting -$\hbar/2$? (1 pt)

g) If one measures S_x, what is the probability of getting $+\hbar/2$? (2 pts)

h) What is the expectation value of S_y (1 pt)

a) the commutation relation

of $[S_x, S_y] = i\hbar \epsilon_{xyz} S_z$

So, S_x, S_y, S_z doesn't commute, so we cannot measure it simultaneously

b) Yes $[S^2, S_z] = 0$

c) $S_z^\dagger = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = S_z$

d) $\begin{vmatrix} \frac{\hbar}{2} - \lambda & 0 \\ 0 & -\frac{\hbar}{2} - \lambda \end{vmatrix} = 0 \Rightarrow (\frac{\hbar}{2} - \lambda)(\frac{\hbar}{2} + \lambda)$

$\Rightarrow \lambda = -\frac{\hbar}{2}, \frac{\hbar}{2}$

- $|S_z = \frac{\hbar}{2}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
- $|S_z = -\frac{\hbar}{2}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

e) $\langle \chi | \chi \rangle = 1$

$\Rightarrow A^2 (1-i \quad 2) \begin{pmatrix} 1+i \\ 2 \end{pmatrix} = 1$

$\Rightarrow A^2 \{ (1-i)(1+i) + 4 \} = 1$

$A^2 (1+1+4) = 1$

$A = \frac{1}{\sqrt{6}}$

$$\chi = \frac{1}{\sqrt{6}}\begin{pmatrix} 1+i \\ 2 \end{pmatrix}$$

f) $P(S_z = +\frac{1}{2}) = |\langle S_z = +\frac{1}{2} | \chi \rangle|^2$

$$= \left| \frac{1}{\sqrt{6}} (1 \quad 0) \begin{pmatrix} 1+i \\ 2 \end{pmatrix} \right|^2$$

$$= \left| \frac{1}{\sqrt{6}} (1+i) \right|^2$$

$$= \frac{1}{6} \sqrt{2} \quad = \frac{1}{3\sqrt{2}}$$

g) $P(S_z = -\frac{1}{2}) = |\langle S_z = -\frac{1}{2} | \chi \rangle|^2$

$$= \left| \frac{1}{\sqrt{6}} (0 \quad 1) \begin{pmatrix} 1+i \\ 2 \end{pmatrix} \right|^2$$

$$= \left| \frac{1}{\sqrt{6}} 2 \right|^2 = \frac{4}{6} = \frac{2}{3}$$

h) $\langle S_y \rangle = \langle \chi | S_y | \chi \rangle$

$$= \frac{1}{6} (1-i \quad 2) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 1+i \\ 2 \end{pmatrix}$$

$$= \frac{\hbar}{12} (1-i \quad 2) \begin{pmatrix} -2i \\ i-1 \end{pmatrix} = \frac{\hbar}{12} \left(-2i + 2i^2 + 2i - 2 \right)$$

$$= -\frac{\hbar}{3}$$

Jan 2009

Problem 2: A two-state system (10 points)

5 09

We can approximate the ammonia molecule NH_3 by a simple two-state system. The three H nuclei are in a plane, and the N nucleus is at a fixed distance either above or below the plane of the H's. Each is approximately a stationary state with some energy E_0. But there is a small amplitude for transition from up to down. Thus the total Hamiltonian is $H = H_0 + H_1$, where

$$H_0 = \begin{pmatrix} E_0 & 0 \\ 0 & E_0 \end{pmatrix} \quad \text{and} \quad H_1 = \begin{pmatrix} 0 & -A \\ -A & 0 \end{pmatrix}$$

with $|A| \ll |E_0|$.

(a) Find the exact eigenvalues of H. (*1 points*)

(b) Now suppose the molecule is in an electric field that distinguishes the two states. The new Hamiltonian is $H = H_0 + H_1 + H_2$, where

$$H_2 = \begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix}$$

Find the new exact energy levels. (*1 points*)

have to solve

(c) Apply perturbation theory and find the energy levels to the lowest non-vanishing order for $\epsilon_i \ll |A|$. Compare the results to the exact answer in (b). (*4 points*)

(d) Apply perturbation theory and find the energy levels to the lowest non-vanishing order for $\epsilon_i \gg |A|$. Compare the results to the exact answer in (b). (*4 points*)

↓ take the exact $H = H_0 + H_2$ & H_1 be the perturbation ?

P2.

$$H = H_0 + H_1 = \begin{pmatrix} E_0 & -A \\ -A & E_0 \end{pmatrix}$$

$$\begin{vmatrix} E_0-\lambda & -A \\ -A & E_0-\lambda \end{vmatrix} = 0 \quad \Rightarrow (E_0-\lambda)^2 - A^2 = 0$$

$$\Rightarrow (E_0-\lambda-A)(E_0-\lambda+A) = 0$$

$$\Rightarrow \lambda = \begin{matrix} E_0 - A \\ E_0 + A \end{matrix}$$

b)

$$H = H_0 + H_1 + H_2$$

$$H = \begin{pmatrix} E_0+\epsilon_1 & -A \\ -A & E_0+\epsilon_2 \end{pmatrix}$$

$$\begin{vmatrix} E_0+\epsilon_1-\lambda & -A \\ -A & E_0+\epsilon_2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (E_0+\epsilon_1-\lambda)(E_0+\epsilon_2-\lambda) - A^2 = 0$$

$$\Rightarrow \{(E_0-\lambda)+\epsilon_1\}\{(E_0-\lambda)+\epsilon_2\} - A^2 = 0$$

$$\Rightarrow (E_0-\lambda)^2 + (E_0-\lambda)(\epsilon_1+\epsilon_2) + \epsilon_1\epsilon_2 - A^2 = 0$$

$$= (E_0-\lambda+A)(E_0-\lambda-A)$$

$$E_0^2 + E_0\epsilon_2 - \lambda E_0 + \epsilon_1 E_0 + \epsilon_1\epsilon_2 - \lambda\epsilon_1 - \lambda E_0 - \lambda\epsilon_2 + \lambda^2 - A^2 = 0$$

$$\Rightarrow \lambda^2 - \lambda\underbrace{(2E_0+\epsilon_1+\epsilon_2)}_{b} + \underbrace{(E_0^2 + E_0\epsilon_2 + \epsilon_1 E_0 + \epsilon_1\epsilon_2 - A^2)}_{c} = 0$$

$$b^2 = \left(\{2E_0 + \epsilon_1 + \epsilon_2\}\right)^2 = \cancel{4E_0^2} + \epsilon_1^2 + \epsilon_2^2 + \cancel{4E_0\epsilon_1} + 2\epsilon_1\epsilon_2 + \cancel{4E_0\epsilon_2}$$

$$- \quad 4ac = \cancel{4E_0^2} + \cancel{4E_0\epsilon_2} + \cancel{4E_0\epsilon_1} + 4\epsilon_1\epsilon_2 - 4A^2$$

$$b^2 - 4ac = \epsilon_1^2 + \epsilon_2^2 - 2\epsilon_1\epsilon_2 + 4A^2$$

$$= (\epsilon - \epsilon_2)^2 + 4A^2$$

$$\lambda = \frac{+(2E_0 + \epsilon_1 + \epsilon_2) \pm \sqrt{(\epsilon_1 - \epsilon_2)^2 + 4A^2}}{2}$$

c) Since $\epsilon_i \ll |A|$

consider, $H = H_0 + H_1$ to be total Hamiltonian & H_2 to be perturbed

then the first ordere energy is $E_n^{(1)} = \langle n^{(0)} | H_2 | n^{(0)} \rangle$

We know, the eigenvalues of $H = H_0 + H_1$ Lets find the eigenvectors

- $\lambda = E_0 - A$

$$\begin{pmatrix} A & -A \\ -A & A \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0 \quad \left.\begin{array}{l} \Rightarrow Aa_1 - Aa_2 = 0 \\ \Rightarrow -Aa_1 + Aa_2 = 0 \end{array}\right\} \Rightarrow a_1 = a_2$$

$$|\lambda = E_0 - A\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

• $\underline{\lambda = E_0 + A}$

$$\begin{pmatrix} -A & -A \\ -A & -A \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow -Ab_1 - Ab_2 = 0$$

$$\Rightarrow b_2 = -b$$

$$|\lambda = E_0 + A\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ - \end{pmatrix}.$$

$$|n^{(0)}\rangle = |\lambda = E_0 - A\rangle + |\lambda = E_0 + A\rangle$$

Thus,

$$E^{(1)} = \langle n^{(0)} | H_2 | n^{(0)} \rangle$$

$$= \langle \lambda = E_0 - A | H_2 | \lambda = E_0 - A\rangle \quad \Big| \quad \langle \lambda = E_0 + A | H_2 | \lambda = E_0 + A \rangle$$

$$= \underset{1\times 2}{\frac{1}{2}(1 \;\; 1)} \underset{2\times 2}{\begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix}} \underset{2\times 1}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} \quad \Big| \quad = \underset{1\times 2}{\frac{1}{2}(1 \;\; -1)} \underset{2\times 2}{\begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix}} \underset{2\times 1}{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

$$= \frac{1}{2}(1 \;\; 1) \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} \quad \Big| \quad = \frac{1}{2}(1 \;\; -1) \begin{pmatrix} \epsilon_1 \\ -\epsilon_2 \end{pmatrix}$$

$$= \frac{1}{2}(\epsilon_1 + \epsilon_2) \quad \Big| \quad = \frac{1}{2}(\epsilon_1 + \epsilon_2)$$

$$E_n = E_0 - A + \frac{1}{2}(\epsilon_1 + \epsilon_2)$$

$$E_n = E_0 + A + \frac{1}{2}(\epsilon_1 + \epsilon_2)$$

Problem 3: 2-d potential (10 points)

Jan 2009 3

A particle of mass m is confined by two impenetrable parallel walls at $x = \pm a$ to move on a two-dimensional strip defined by

$$-a < x < a$$
$$-\infty < y < \infty$$

The wave function for this system can be expressed as the product of two functions: one that depends only on the spatial co-ordinates (x and y), and one that depends only on time t.

a) Use the separation of variables technique to find the time dependent function. (2 points)

b) The part of the wave function that depends only on spatial co-ordinates can be expressed as the product of two functions: one that depends only on x and one that depends only on y. Use the separation of variables technique to find these two functions. (3 points)

c) What is the minimum energy of the particle that measurement can yield? (2 points)

d) Suppose that two additional walls are inserted at $y = \pm a$. Can a measurement of the particle's energy yield the value $3\pi^2\hbar^2/8ma^2$ Explain your answer. (3 points)

P3

$$H\,\psi(x,y,t) = E\,\psi(x,y,t)$$

$$\Rightarrow \quad i\hbar \frac{\partial}{\partial t}\psi(x,y,t) = -\frac{\hbar^2}{2m}\nabla^2\psi(x,y,t) + V(x,y)\,\psi(x,y,t)$$

$$\psi(x,y,t) = \phi(x,y)\,f(t)$$

$$\Rightarrow \quad i\hbar\,\phi(x,y)\frac{df(t)}{dt} = -\frac{\hbar^2}{2m}f(t)\,\nabla^2\phi(x,y) + V(x,y)\,\phi(x,y)\,f(t)$$

divide by $\phi(x,y)f(t)$ $\qquad$ if $V(x,y) = 0$

$$\Rightarrow \quad i\hbar\frac{1}{f(t)}\frac{df(t)}{dt} = -\frac{\hbar^2}{2m}\frac{1}{\phi(x,y)}\nabla^2\phi(x,y) + \cancel{V(x,y)}^{\,0}$$

LHS & RHS are independent to each other so they has to be equal to a const. which in this case has to to be the total energy E

$$i\hbar\frac{1}{f(t)}\frac{df(t)}{dt} = E$$

$$\Rightarrow \int_{t_0}^{t}\frac{df}{f} = \int_{t_0}^{t} -iE/\hbar \, dt'$$

$$\Rightarrow f(t) = e^{-\frac{iE(t-t_0)}{\hbar}}$$

Now,

$$-\frac{\hbar^2}{2m}\nabla^2\phi(x,y) + \cancel{V(x,y)}^{\,0}\,\phi(x,y) = E\,\phi(x,y)$$

$$\left.\begin{aligned}\phi(x,y) &= \psi(x)\,\psi(y)\\ \times V(x,y) &= V(x)\,V(y)\end{aligned}\right\}$$

$$\Rightarrow -\frac{\hbar^2}{2m}\left\{\psi(y)\frac{d^2\psi(x)}{dx^2} + \psi(x)\frac{d^2\psi(y)}{dy^2}\right\} + \cancel{V(x,y)}^{\,0} = E\,\psi(x)\psi(y)$$

$$\Rightarrow -\frac{\hbar^2}{2m}\left\{\frac{1}{\psi(x)}\frac{d^2\psi(x)}{dx^2} + \frac{1}{\psi(y)}\frac{d^2\psi(y)}{dy^2}\right\} = \underset{E_x+E_y}{E}$$

Let,

$$-\frac{\hbar^2}{2m}\frac{1}{\psi(x)}\frac{d^2\psi(x)}{dx^2} = \underset{E_x}{l} \Rightarrow \frac{d^2\psi(x)}{dx^2} = -\frac{\overbrace{2ml}^{k^2}}{\hbar^2}\psi(x)$$

$$\Rightarrow \lambda^2 = -k^2$$

$$\Rightarrow \psi(x) = A\cos kx + B\sin kx$$

then

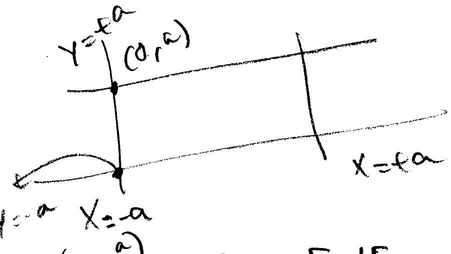
$$l - \frac{\hbar^2}{2m}\frac{1}{\psi(y)}\frac{d^2\psi(y)}{dy^2} = \underline{\underline{E}}$$

$$\Rightarrow -\frac{\hbar^2}{2m}\frac{1}{\psi(y)}\frac{d^2\psi(y)}{dy^2} = \underbrace{E-l}_{E-E_x=E_y}$$

$E_x+E_y=E$

$$\Rightarrow \frac{d^2\psi(y)}{dy^2} = -\frac{2m}{\hbar^2}(E-l)\,\psi(y)$$

$$= \underbrace{\frac{2m}{\hbar^2}(l-E)}_{\beta}\psi(y)$$



$$\Rightarrow \psi(y) = Ce^{\beta y} + De^{-\beta y} \ne 0$$

$$= C\cos kx + D\sin ky \qquad e^{-inx} + e^{+inx}$$

$E = E_x + E_y$

$\psi_I(x=0) = \psi_{II}(x=0)$

$\Rightarrow A = 0$

$\psi_I(x=a) = \psi_{III}(x=a)$

$\Rightarrow B \sin ka = 0$

$\Rightarrow k = \dfrac{n\pi}{a}$

$\Rightarrow$ Now, $k^2 = \dfrac{2ml}{\hbar^2}$

$\Rightarrow \dfrac{n^2\pi^2}{a^2} = \dfrac{2ml}{\hbar^2} \Rightarrow l = \dfrac{n^2\pi^2\hbar^2}{2ma^2}$

$\psi_I(x=-a) = 0$

$A\cos ka - B\sin ka = 0$

$\psi_{II}(x=a) = 0$

$\Rightarrow A\cos ka + B\sin ka = 0$

$2A\cos ka = 0$

$\Rightarrow k = \dfrac{n\pi}{2a} \Rightarrow k^2 = \dfrac{2ml}{\hbar^2}$

$\Rightarrow \dfrac{n^2\pi^2}{4a^2} = \dfrac{2ml}{\hbar^2}$

$l = \dfrac{n^2\pi^2\hbar^2}{8ma^2}$

$E_{n_x} = \dfrac{n_x^2\pi^2\hbar^2}{8ma^2}$

$E_{1x} = \dfrac{\pi^2\hbar^2}{8ma^2}$

$E_{1y} = 0$

$$\frac{1}{\psi(x)}\frac{d^2\psi(x)}{dx^2} + \frac{1}{\psi(y)}\frac{d^2\psi(y)}{dy^2} = -\frac{2mE}{\hbar^2}$$

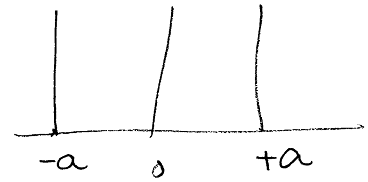
$$\frac{d^2\psi(x)}{dx^2} = l\,\psi(x) \Rightarrow \psi_I(x) = Ae^{lx} + Be^{-lx}$$

$$\frac{d^2\psi(y)}{dy^2} = -\frac{2m(E+l)}{\hbar^2}\psi(y)$$

$$\frac{d^2\psi(y)}{dy^2} = -k^2\psi(y) \Rightarrow \lambda = \pm ik$$

$$\psi_I(y) = C\cos kx + D\cos ky$$

$$\psi_I(x) = A e^{lx} + B e^{-lx}$$



$$\psi_I(y) = C \cos kx + D \sin kx$$

$$\psi_I(x=-a) = 0$$

$$\Rightarrow A e^{-la} + B e^{+la} = 0$$

$$\left.\frac{d\psi_I(x)}{dx}\right|_{x=-a} = 0$$

$$\Rightarrow l A e^{-la} - B e^{la} = 0$$

$$\psi_I(x=a) = 0$$

$$A e^{la} + B e^{-la} = 0$$

$$\left.\frac{d\psi_I(x)}{dx}\right|_{x=a} = 0$$

$$\Rightarrow l A e^{la} - B e^{-la} = 0$$

$$\psi_I(y=-a) = C e^{-a\beta} + D e^{a\beta} = 0$$

$$\Rightarrow$$

$$\psi_{II}(y=a) = C e^{-a\beta}$$

Jan 2009

Problem 4: Angular momentum (10 points)

A $|jm\rangle = |1,0\rangle$ state scatters from a $|jm\rangle = |\frac{1}{2},\frac{1}{2}\rangle$ state via a $|jm\rangle = |\frac{1}{2},\frac{1}{2}\rangle$ resonance.

a) Relate the highest weight (highest possible m) states in the total j basis to the highest weight states in the direct product basis for this system of $\frac{1}{2} \otimes 1$. (1 pt)

b) Acting on the highest weight states with lowering operators, give an expansion of each total-j state in terms of direct product states and their Clebsch-Gordon co-efficients. (5 pts) *Hint*: $J_{\pm}|jm\rangle = \hbar[(j \mp m)(j \pm m + 1)]^{1/2}|j, m \pm 1\rangle$

c) How often do the above-mentioned spin states scatter elastically, and how often do they scatter inelastically? (4 pts)

no idea

P4 $j_1 = 1 \quad m_{j_1} = 1, 0, -1$ $\qquad J = j_1 + j_2 = \frac{3}{2}, \frac{1}{2}$

a) $j_2 = \frac{1}{2} \quad m_{j_2} = \frac{1}{2}, -\frac{1}{2}$ $\qquad m_J = \frac{3}{2}, \frac{1}{2}, 0, \frac{1}{2}, \frac{3}{2}$

$$|J=\tfrac{3}{2}, M=\tfrac{3}{2}\rangle = |j_1=1, m_j=1\rangle \otimes |j_2=\tfrac{1}{2}, m_j=\tfrac{1}{2}\rangle$$

$$= |1,1\rangle \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle$$

$$= |1,\tfrac{1}{2}; 1,\tfrac{1}{2}\rangle$$

b) $J_- |J=\frac{3}{2}, M=\frac{3}{2}\rangle = (J_{1-} + J_{2-}) |1,\frac{1}{2}; 1,\frac{1}{2}\rangle$

$$\Rightarrow \hbar\sqrt{\underbrace{\tfrac{3}{2}(\tfrac{3}{2}+1)}_{\frac{15}{4}} - \tfrac{1}{2}(\tfrac{1}{2}-1)}\;|J=\tfrac{3}{2}, M=\tfrac{1}{2}\rangle = \hbar\left[\sqrt{\underbrace{1(1+1)}_{2} - \underbrace{1(1-1)}_{=0}}\;|1,\tfrac{1}{2};0,\tfrac{1}{2}\rangle + \sqrt{\underbrace{\tfrac{1}{2}(\tfrac{1}{2}+1)}_{3/4} - \underbrace{\tfrac{1}{2}(\tfrac{1}{2}-1)}_{-\frac{1}{4}}}\;|1,\tfrac{1}{2};1,-\tfrac{1}{2}\rangle\right]$$

$\frac{15}{4} + \frac{1}{4}$

$$\Rightarrow 2\hbar\,|J=\tfrac{3}{2}, M=\tfrac{1}{2}\rangle = \sqrt{2}\,\hbar\,|1,\tfrac{1}{2};0,\tfrac{1}{2}\rangle + \hbar\,|1,\tfrac{1}{2};1,-\tfrac{1}{2}\rangle$$

$$\Rightarrow |J=\tfrac{3}{2}, M=\tfrac{1}{2}\rangle = \frac{1}{\sqrt{2}}|1,\tfrac{1}{2};0,\tfrac{1}{2}\rangle + \frac{1}{2}|1,\tfrac{1}{2};1,-\tfrac{1}{2}\rangle$$

$$\rightsquigarrow |J=\tfrac{1}{2}, M=\tfrac{1}{2}\rangle = \alpha\,|1,\tfrac{1}{2};0,\tfrac{1}{2}\rangle + \beta\,|1,\tfrac{1}{2};1,-\tfrac{1}{2}\rangle$$

$$\langle J=\tfrac{1}{2}, M=\tfrac{1}{2} | J=\tfrac{1}{2}, M=\tfrac{1}{2}\rangle = \alpha^2 + \beta^2$$

$$\langle J=\tfrac{3}{2}\ M=\tfrac{1}{2} \mid J=\tfrac{1}{2}\ M=\tfrac{1}{2}\rangle = 0$$

$$\frac{\alpha}{\sqrt{2}} + \frac{\beta}{2} = 0$$

$$\Rightarrow \alpha = -\frac{\beta}{\sqrt{2}}$$

$$\rightsquigarrow (-\beta/\sqrt{2})^2 + \beta^2 = 1$$

$$\Rightarrow \frac{3\beta^2}{2} = 1 \Rightarrow \beta = \pm\sqrt{\frac{2}{3}} \Rightarrow \alpha = \mp\frac{1}{\sqrt{3}}$$

choose, $\beta = +\sqrt{\frac{2}{3}}$ since $m_1 = 1$ is max

$$\alpha = -\frac{1}{\sqrt{3}}$$

$$|J=\tfrac{1}{2}, M=\tfrac{1}{2}\rangle = \sqrt{\frac{2}{3}}\,|1,\tfrac{1}{2};1,-\tfrac{1}{2}\rangle - \frac{1}{\sqrt{3}}\,|1,\tfrac{1}{2};0,\tfrac{1}{2}\rangle$$

$\frac{3}{2}, \frac{1}{2}$ $\frac{3}{2}, \frac{1}{2}$

Problem 5: Measurement and Probability (10 points) 5 Jan 2009

Consider the following two observables, H and C, whose representation in the unit basis $|e_1\rangle$, $|e_2\rangle$ and $|e_3\rangle$ is:

$$H = \hbar\omega \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, C = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

where:

$$|e_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, |e_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, |e_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Assume that at time t=0 the ensemble of particles is in the state:

$$|\Psi(0)\rangle = \frac{1}{\sqrt{2}}|e_1\rangle + \frac{1}{\sqrt{2}}|e_2\rangle$$

The eigenvalues of H are given by $\lambda = 2, 1, -1$ with normalized eigenvectors given by $(1,1,1)/\sqrt{3}$, $(1,0,-1)/\sqrt{2}$ and $(1,-2,1)/\sqrt{6}$ respectively.

The eigenvalues of C are given by $\lambda = 1, 1, -1$ with normalized eigenvectors given by $(1,0,-1)/\sqrt{2}$, $(0,1,0)$ and $(1,0,1)/\sqrt{2}$ respectively.

a) What is the probability of measuring H and obtaining $E = \hbar\omega$? What state is the particle in after the measurement? (2 pts)

b) If one immediately measures C after the measurement of H in part b), what is the probability of obtaining $c = 1$? (1 pt)

c) What is the probability of measuring H first and getting $E = \hbar\omega$, then measuring C and getting $c = 1$, i.e. what is $P_{|\Psi(0)\rangle}(E = \hbar\omega, c = 1)$? (1 pt)

d) If the system is allowed to evolve in time after the measurement of H and before C is measured, will your answer to part c) change? Explain your reasoning. (1 pt)

e) With the ensemble of particles all in the original state: $|\Psi(0)\rangle = \frac{1}{\sqrt{2}}|e_1\rangle + \frac{1}{\sqrt{2}}|e_2\rangle$, reverse the order of the above measurements and answer the same questions:

i) What is the probability of obtaining $c = 1$ if C is measured first? What state is the particle in after C is measured? (1 pt)

ii) If one immediately measures H after C is measured in part i), what is the probability of obtaining $E = \hbar\omega$? (1 pt) **(question continues on next page...)**

iii) What is the composite probability $P_{|\Psi(0)\rangle}(c = 1, E = \hbar\omega)$? (1 pt)

iv) If the system had been allowed to evolve in time after the measurement of C and before H is measured, would your answer to part ii) be different? Explain. (1 pt)

f) Are H and C compatible observables? Why?

P5 $\quad |\Psi(0)\rangle = \frac{1}{\sqrt{2}}|e_1\rangle + \frac{1}{\sqrt{2}}|e_2\rangle$

a) $|E=\hbar\omega\rangle = \frac{1}{\sqrt{2}}|e_1\rangle - \frac{1}{\sqrt{2}}|e_3\rangle$

$$P(E=\hbar\omega) = |\langle E=\hbar\omega|\Psi(0)\rangle|^2$$
$$= \frac{1}{4}$$

$$|\Psi(0)\rangle = \frac{1}{\sqrt{2}}|e_1\rangle$$

b) $|c=1,1\rangle = \frac{1}{\sqrt{2}}|e_1\rangle - \frac{1}{\sqrt{2}}|e_3\rangle \qquad |c=1,2\rangle = |e_2\rangle$

$$P(c=1) = |\langle c=1,1|\Psi(0)\rangle|^2 + |\langle c=1,2|\Psi(0)\rangle|^2$$
$$= 1 + 0$$

c) $P_{|\psi(0)\rangle}(E=\hbar\omega, c=1) = P(E=\hbar\omega)\,P(c=1)$
$$= \frac{1}{4} \times 1$$
$$= \frac{1}{4}$$

d) NO, Since the state collapse in the eigenstate of the Hamiltonian, which is a stationary state

also, if $[C,H]=0$ we can say C would be const. of motion

?

$$HC = \begin{pmatrix} 0 & 1 & -1 \\ -1 & & \\ & & \end{pmatrix}$$

e) $|\Psi(0)\rangle = \frac{1}{\sqrt{2}}|e_1\rangle + \frac{1}{\sqrt{2}}|e_2\rangle$

$|C=1,1\rangle = \frac{1}{\sqrt{2}}|e_1\rangle - \frac{1}{\sqrt{2}}|e_3\rangle$

$|C=1,2\rangle = |e_2\rangle$

i) $P(c=1) = |\langle C=1|\Psi(0)\rangle|^2$

$= |\langle c=1,1|\Psi(0)\rangle|^2 + |\langle c=1,2|\Psi(0)\rangle|^2$

$= |\frac{1}{\cancel{\sqrt{}}2}|^2 + |\frac{1}{\cancel{\sqrt{}}2}|^2$

$= \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$

$|E=\hbar\omega\rangle = \frac{1}{\sqrt{2}}[|e_1\rangle - |e_3\rangle]$ its in the $|C=1\rangle$ state

ii) $P(E=\hbar\omega) = |\langle E=\hbar\omega|C=1\rangle|^2$

$= |\langle E=\hbar\omega|C=1,1\rangle|^2 + |\langle E=\hbar\omega|C=1,2\rangle|^2$

$= |\frac{1}{2}+\frac{1}{2}|^2 + 0$

$= 1$

iii) $P_{|\Psi(0)\rangle}(C=1, E=\hbar\omega) = P(C=1)\,P(E=\hbar\omega)$

$= \frac{3}{4}$

iv) its not different as $E=\langle H\rangle$ is a const of motion

f) $[C,H] \neq 0$ so they are not compatible

Problem 6: The hydrogen atom (10 points)

7 Jan 2009

5-09

The figure below shows the radial function $R_{n,\ell}(r)$ for a stationary state of atomic hydrogen. The normalized Hamiltonian eigenfunction for this state, in atomic units, is

$$\psi_{n,\ell,m_\ell}(\boldsymbol{r}) = \frac{1}{81}\sqrt{\frac{2}{\pi}}(6-r)\, e^{-r/3}\, \cos\theta. \tag{1}$$

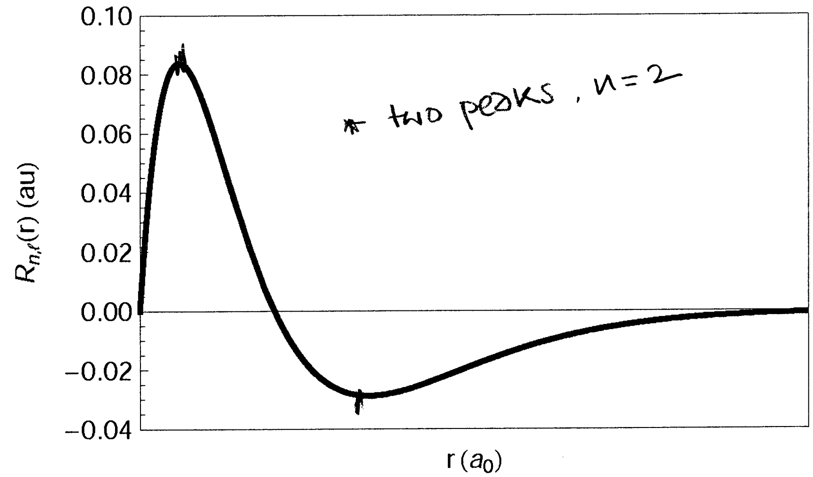


Figure 1: A radial function for a stationary state of atomic hydrogen.

1. **3 points.** What are the values of the quantum numbers n, ℓ, and m_ℓ for this state? To receive any credit, you must fully justify your answer.
2. **1 points.** What is the energy (in eV) of this state?
3. **2 points.** What are the mean value and uncertainty in r (in atomic units) for this state?
4. ? **2 points.** Calculate the value of r (in atomic units) at which a position measurement would be most likely to find the electron if the atom is in this state. $\frac{\partial}{\partial r}(\psi^*\psi) \overset{?}{=} 0$
5. **2 points.** From Eq. 1, generate the normalized eigenfunction $\psi_{n,\ell,m_\ell+1}(\boldsymbol{r})$.

Hint:

$$\int_0^\infty e^{-2r/3} r^n \,\mathrm{d}r = n!\left(\frac{3}{2}\right)^{n+1} \tag{2}$$

Hint: *The following table gives the orbital-angular-momentum operators in Cartesian and spherical coordinates.*

Component	Cartesian coordinates	Spherical coordinates
$\widehat{L}_x$	$-\imath\hbar\left(y\dfrac{\partial}{\partial z}-z\dfrac{\partial}{\partial y}\right)$	$\imath\hbar\left(\sin\varphi\dfrac{\partial}{\partial\theta}+\cot\theta\cos\varphi\,\dfrac{\partial}{\partial\varphi}\right)$
$\widehat{L}_y$	$-\imath\hbar\left(z\dfrac{\partial}{\partial x}-x\dfrac{\partial}{\partial z}\right)$	$-\imath\hbar\left(\cos\varphi\dfrac{\partial}{\partial\theta}-\cot\theta\sin\varphi\,\dfrac{\partial}{\partial\varphi}\right)$
$\widehat{L}_z$	$-\imath\hbar\left(x\dfrac{\partial}{\partial y}-y\dfrac{\partial}{\partial x}\right)$	$-\imath\hbar\dfrac{\partial}{\partial\varphi}$
$\widehat{\boldsymbol{L}}^2$	$\widehat{L}_x^2+\widehat{L}_y^2+\widehat{L}_z^2$	$-\hbar^2\left[\dfrac{1}{\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial}{\partial\theta}\right)+\dfrac{1}{\sin^2\theta}\dfrac{\partial^2}{\partial\varphi^2}\right]$

Table 1: Components and square of the orbital angular momentum operator in Cartesian and spherical coordinates.

P6

1. $$\psi_{n,l,m}(r) = \frac{1}{81}\sqrt{\frac{2}{\pi}}\,(6-r)\,e^{-r/3}\cos\theta$$

$$\hat{L}^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

$$\hat{L}^2\,\psi_{nlm}(r) = \frac{-\hbar^2}{81}\sqrt{\frac{2}{\pi}}\,(6-r)\,e^{-r/3}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(-\sin^2\theta\right)\right]$$

$$\left[\frac{-1}{\sin\theta}\,2\sin\theta\cos\theta\right]$$

$$= \frac{2\hbar^2}{81}\sqrt{\frac{2}{\pi}}\,(6-r)\,e^{-r/3}\cos\theta$$

$$= 2\hbar^2\,\psi_{nlm}(r)$$

$$\hbar^2\,l(l+1) = 2\hbar^2 \Rightarrow l(l+1) = 2 \Rightarrow l^2 + l - 2 = 0$$

$$\Rightarrow l = \frac{-1 \pm \sqrt{1+8}}{2} = \frac{-1\pm 3}{2} = -2,\ 1$$

$l \neq -2$ since nonphysical

$l = 1$

$$\hat{L}_z\,\psi_{nlm}(r) = 0\,\psi_{n,l,m}(r) \Rightarrow m = 0$$

Hence, $|n\,l\,m_l\rangle = |2\,1\,0\rangle$

2. $E_n = \frac{-13.6}{n^2} = -\frac{13.6}{4}$

3. $\langle r \rangle = \int \psi^*_{nlm}(r)\, r\, \psi_{nlm}(r)\, r^2 dr \sin\theta\, d\theta\, d\phi$

$\Rightarrow \langle r \rangle = \frac{2\pi}{(81)^2} \frac{2}{\pi} \int (6-r)^2 e^{-2r/3} r^3 dr \underbrace{\int_0^\pi \cos^2\theta \sin\theta d\theta}_{\frac{2}{3}}$

$= \frac{\cancel{8}\,4}{(81)^2} \left[\int_0^\infty 6^2 r^3 e^{-2r/3} dr - 12 \int_0^\infty e^{-2r/3} r^4 dr + \int e^{-2r/3} r^5 dr \right]$

$= \frac{8}{(81)^2} \times \frac{2}{3} \left[6^2\, 3! \left(\frac{3}{2}\right)^4 - 12 \cdot 4! \left(\frac{3}{2}\right)^5 + 5! \left(\frac{3}{2}\right)^6 \right]$

$= \frac{8}{(81)^2} \times \frac{2}{3} \left[\frac{3^6}{2^2} - \underbrace{4\times3\times4\times2 \left(\frac{3}{2}\right)^5}_{3^6} + \underbrace{5\times4\times3\times2 \left(\frac{3}{2}\right)^6}_{\frac{3^7\times5}{2^3}} \right]$

$= \frac{2^4}{3^5} \left[\frac{3^6}{2^2} - 3^6 + \frac{3^7\times5}{2^3} \right]$

$= 3\times4 - 2^4\times3 + 3^2\times5\times2$

$= 12 - 48 + 90$

$= 54$

$$\langle r^2\rangle = \frac{2\cancel{\pi}}{(81)^2}\,\frac{2}{\cancel{\pi}}\times\frac{2}{3}\left[\int_0^\infty 6^2 r^4 e^{-2r/3}\,dr - 12\int_0^\infty r^5 e^{-2r/3}\,dr + \int e^{-2r/3} r^6\,dr\right]$$

$$= \frac{2^3}{3^5}\left[2^2\cdot 3^2\ 4!\left(\frac{3}{2}\right)^5 - \underbrace{2^2\times 3\times 5!\left(\frac{3}{2}\right)^6} + \cancel{6}!\left(\frac{3}{2}\right)^7\right]$$

$$= \underbrace{\frac{2^8}{3^5}\times 3^3\left(\frac{3}{2}\right)^5}_{=\,2^3\times 3^3} - \underbrace{\frac{2^3}{3^5}\times 2^2\times 3\times 5\times 2^2\times 3\left(\frac{3}{2}\right)^6}_{\underbrace{-\frac{2^7\times 3^2\times 5\times 3^6}{3^5\times 2^6}}_{=\,2\times 3^3\times 5}} + \underbrace{6\times 5\times 4\times 3\times 2\left(\frac{3}{2}\right)^7}_{\underbrace{2\times 3\times 5\times 2^2\times 3\times 2\left(\frac{3}{2}\right)^7}_{\underbrace{\frac{2^4\times 3^2\times 5\times 3^7}{2^7}}_{=\,\frac{3^9\times 5}{2^3}}}}$$

$$= 2^3\times 3^3 - 2\times 3^3\times 5 + \frac{3^9\times 5}{2^3}$$

$$= 3^3\times 2\left[2 - 10 + \frac{27\times 5}{16}\right]$$

$$= 27\times 2\left[\frac{32-160+135}{16}\right]$$

$$= \frac{27}{8}(32-25) = \frac{27\times 7}{8}$$

Not working.

4. ?

$$P(r) = \frac{1}{(81)^2}\frac{2}{\pi}(6-r)^2 e^{-2r/3} r^2 \underbrace{\int_0^{\pi} \cos^2\theta \sin\theta d\theta \int_0^{2\pi} d\phi}_{\frac{2}{3}\times 2\pi}$$

$$= \frac{8}{3(81)^2}\left[r^2 (6-r)^2 e^{-2r/3}\right]$$

$$\frac{dP(r)}{dr} = 0$$

$$\Rightarrow \frac{8}{3(81)^2}\left[2r(6-r)^2 e^{-2r/3} - 2r^2(6-r)e^{-2r/3} - \frac{2}{3}r^2(6-r)^2 e^{-2r/3}\right] = 0$$

$$\Rightarrow \left[6-r-r-\frac{2}{3}r(6-r)\right] = 0$$

$$\Rightarrow \frac{1}{3}r(6-r)+2r-6=0$$

$$\Rightarrow 2r-\frac{1}{3}r^2+2r-6=0$$

$$\Rightarrow 4r-\frac{1}{3}r^2-6=0$$

$$\Rightarrow \frac{1}{3}r^2-4r+6=0$$

$$\Rightarrow r^2-12r+18=0$$

$$\Rightarrow r = \frac{12\pm\sqrt{144-72}}{2} = \frac{12\pm\sqrt{72}}{2}$$

$$= 6\pm$$

5.

$$\hat{L}_+ = \hat{L}_x + i\hat{L}_y$$

$$= i\hbar\left(\sin\phi \frac{\partial}{\partial\theta} + \cot\theta \cos\phi \frac{\partial}{\partial\phi}\right) + \hbar\left(\cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right)$$

$$\hat{L}_+ \psi_{n,l,m} = i\hbar \sin\phi \frac{\partial \psi_{n,l,m}}{\partial\theta} + \hbar\cos\phi \frac{\partial \psi_{n,l,m}}{\partial\theta}$$

$$\Rightarrow \psi_{n,l,m+1} = \frac{1}{81}\sqrt{\frac{2}{\pi}}\, i\hbar \sin\phi\, (6-r)e^{-r/3}(-\sin\theta)$$

$$+ \hbar\cos\phi\,(6-r)\, e^{-r/3}(-\sin\theta)\,\frac{1}{81}\sqrt{\frac{2}{\pi}}$$

$$= \frac{\hbar}{81}\sqrt{\frac{2}{\pi}}(6-r)e^{-r/3}(-i\sin\phi + \cos\phi)\sin\theta$$

$$= \underbrace{\frac{\hbar}{81}\sqrt{\frac{2}{\pi}}(6-r)e^{-r/3}\sin\theta}\, e^{-i\phi}$$

$$\hat{L}_z \psi_{n,l,m_{l+1}} = -i\hbar\frac{\partial}{\partial\phi}\underbrace{\quad}(-i)\, e^{-i\phi} = \hbar\, \psi_{n,l,m_{l+1}}$$