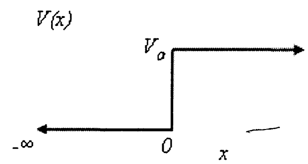
QUANTUM QUALIFIER EXAM, JANUARY 2007

PROBLEM 1



Consider the step potential shown in the figure.

a) [1 pts] Consider a particle traveling from $x = -\infty$ to the right with an energy E. The appropriate wavefunction for this particle is given by

$$\phi = \begin{cases} e^{ik_L x} + Ae^{-ik_L x} & \text{for } x < 0 \\ Be^{ik_R x} & \text{for } x > 0 \end{cases}$$

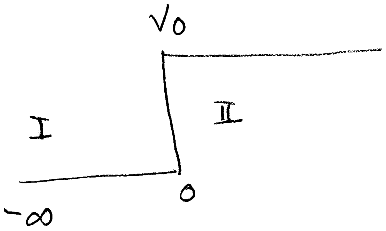
Give expressions for k_L and k_R and define any undefined parameters/constants given in your expression.

b) [3 pts] For the case that $E > V_o$, use appropriate boundary conditions to find the coefficients A and B.

c) [2 pts] For the case that $E > V_o$, find the probability that the particle will be reflected.

d) [2 pts] For the case that $E > V_o$, the probability that the particle will be transmitted is given by $T = 1 - R$. Determine and explain the physical meaning of the ratio $|B|^2/T$.

e) [2 pts] What is the probability for reflection when $E < V_o$?



P1.

a) $\phi = \begin{cases} e^{iK_L x} + A e^{-iK_L x} & x < 0 \\ B e^{iK_R x} & x > 0 \end{cases}$

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right]\psi = E\psi$$

$$K_L = \frac{2mE}{\hbar^2} \qquad K_L = \frac{2m(E-V_0)}{\hbar^2}$$

$$\frac{d^2}{dx^2} = -\frac{2m(E-V_0)}{\hbar^2}$$

b) $\psi_I(x=0) = \psi_{II}(x=0)$

$\Rightarrow 1 + A = B$

$\Rightarrow A = B - 1$

$\Rightarrow A = \frac{K_L - K_R}{K_R}$

$\left.\frac{d\psi_I}{dx}\right|_{x=0} = \left.\frac{d\psi_{II}}{dx}\right|_{x=0}$

$\Rightarrow \left. iK_L e^{iK_L x}\right|_{x=0} = \left. iK_R B e^{iK_R x}\right|_{x=0}$

$\Rightarrow iK_L = iK_R B$

$\Rightarrow B = \frac{K_L}{K_R}$

c) $R = \left|\frac{B}{A}\right|^2 = \left|\frac{K_L}{K_L - K_R}\right|^2 = \left(\frac{K_L}{K_L - K_R}\right)^2$

d) $T = 1 - R = \frac{(K_L - K_R)^2 - K_L^2}{(K_L - K_R)^2} = \frac{K_R^2 - 2K_L K_R}{(K_L - K_R)^2}$

$\frac{|B|^2}{T} = \frac{|B|^2}{1 - \frac{|B|^2}{|A|^2}} = \frac{|A|^2 |B|^2}{|A|^2 - |B|^2}$

$$\frac{|B|^2}{T} = \frac{K_L^2 / K_R^2}{\frac{2K_R K_L}{K_R^2} - \frac{K_L^2}{K_R^2}} = \frac{K_L}{2K_R + K_L}$$

c) $R = \frac{|A|^2}{1} = \left(\frac{K_L - K_R}{K_R}\right)^2$

$$T = 1 - R = \frac{K_R^2 - (K_L - K_R)^2}{K_R^2} = \frac{K_L(2K_R - K_L)}{K_R^2} = \frac{2K_R K_L}{K_R^2} + \frac{K_L^2}{K_R^2}$$

$$J_{trans} = \frac{\hbar}{2im}\left[|B|^2 e^{-ik_R x}(ik)e^{ik_R x} - |B|^2 e^{ik_R x}(-ik_R)e^{ik_R x}\right]$$

$$= \frac{\hbar|B|^2}{2im} 2ik_R = \frac{\hbar k_R |B|^2}{m}$$

$$J_{inc} = \frac{\hbar}{2im}\left[e^{-ik_L x}(ik_L)e^{ik_L x} - e^{ik_L x}(-ik_L)e^{-ik_R x}\right]$$

$$= \frac{\hbar k_L}{m}$$

$$T = \left|\frac{J_{trans}}{J_{inc}}\right|^2 = \frac{k_R^2}{k_L^2}|B|^2$$

$$\Rightarrow \frac{|B|^2}{T} = \frac{k_L^2}{k_R^2} = \frac{E}{(E-V_0)}$$

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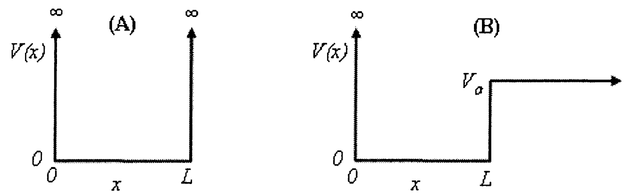
PROBLEM 2



a) [2 pts] Calculate the energy eigenvalues for a particle of mass m in the one-dimensional infinite well shown in Figure A.

b) [4 pts] For the time-independent Schrödinger Equation corresponding to potential (B), find a transcendental equation in E giving the eigenenergies in terms of $V_o, L, m,$ and $\hbar$

c) [4 pts] For the time-independent Schrödinger Equation corresponding to potential (B), what is the smallest value of V_o that gives one bound state? What is the smallest value of V_o that gives two bound states?

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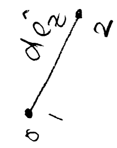


PROBLEM 3

Consider a quantum mechanical system that consists of two identical spin 1/2 particles that are fixed in space, separated by a distance d. Particle 1 is at the origin ($\vec{r}_1 = \vec{0}$) whereas particle 2 is at $\vec{r}_2 = d\,\hat{e}_z$. Each particle has a magnetic moment

$$\vec{\mu}(j) = \frac{g\mu_o}{\hbar}\vec{S}(j), \quad j = 1,2$$

and a $g-$factor $g = 2$. $\vec{S}(j)$ is the spin operator of the j^{th} particle. Throughout this problem we will use the basis states $|1\rangle = |+,+\rangle$, $|2\rangle = |+,-\rangle$, $|3\rangle = |-,+\rangle$, and $|4\rangle = |-,-\rangle$, where these are the usual states written in terms of the $z-$components of the particles' spins.

a) [2pts] First consider what happens if we place the particles in an external magnetic field $\vec{B} = B\hat{e}_z$. Write the matrix representation for the Hamiltonian of the system $H_o = -\vec{\mu}\cdot\vec{B}$ in the $|i\rangle$, $i = 1,2,3,4$ basis given above, considering only the interaction between the spins and the magnetic field. What are the energy eigenstates and eigenvalues for the system? Draw an energy-level diagram.

b) [3pts] We know, however, that the magnetic moment of each particle will create a magnetic field that the other particle will feel. The dipole field from particle 1 at particle 2 is (classically)

$$\vec{B}_{21} = \frac{1}{d^3}(3\mu_z(1)\hat{e}_z - \vec{\mu}(1))$$

so that the interaction Hamiltonian between the two particles is

$$\begin{aligned} \hat{H}' &= -\vec{\mu}_2 \cdot \vec{B}_{21} \\ &= \frac{g^2\mu_o^2}{\hbar^2 d^3}\left(-3S_z(1)S_z(2) + \vec{S}(1)\cdot\vec{S}(2)\right). \end{aligned}$$

Compute the action of the interaction Hamiltonian on each of the basis states. In other words, calculate $\hat{H}'|i>$ for $i = 1,2,3,4$.
Hint: Use the usual angular momentum raising and lowering operators

$$\hat{S}^{\pm} = \hat{S}_x(j) \pm i\hat{S}_y(j), \; j = 1,2$$

c) [2pts] Write the total Hamiltonian, $\hat{H} = \hat{H}_o + \hat{H}'$ as a matrix in the $|i\rangle$ basis.

d) [3pts] Find the eigenstates and eigenvalues of this total Hamiltonian and draw the energy level diagram as a function of the magnetic field strength.

P3

$$\vec{\mu}(j) = \frac{g\mu_0}{\hbar}\vec{S}(j)$$

$$\rightsquigarrow \vec{\mu}_1 = \frac{2\mu_0}{\hbar}\vec{S}_1 \qquad \rightsquigarrow \vec{\mu}_2 = \frac{2\mu_0}{\hbar}\vec{S}_2$$

$$\rightarrow \vec{\mu} = \vec{\mu}_1 + \vec{\mu}_2 = \frac{2\mu_0}{\hbar}(\vec{S}_1 + \vec{S}_2)$$

$S_z|s, m_s\rangle = m_s\hbar|s, m_s\rangle$

Now, $H_0 = -\vec{\mu}\cdot\vec{B} = -\frac{2\mu_0 B}{\hbar}(S_{1z} + S_{2z})$

$$H_0 = \begin{array}{c|cccc} & |++\rangle & |+-\rangle & |-+\rangle & |--\rangle \\ \langle ++| & 1/4 & 0 & 0 & 0 \\ \langle +-| & 0 & 0 & 0 & 0 \\ \langle -+| & 0 & 0 & 0 & 0 \\ \langle --| & 0 & 0 & 0 & -\frac{1}{4} \end{array} \left(-\frac{2\mu_0 B}{\hbar}\right)$$

$$= -\frac{\mu_0 B}{2}\begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & -1 \end{pmatrix} \qquad (S_{1z} + S_{2z})|+-\rangle$$

So, the eigen values are,

$$\lambda = \pm\frac{\mu_0 B}{2}$$

• $\lambda = +\frac{\mu_0 B}{2}$

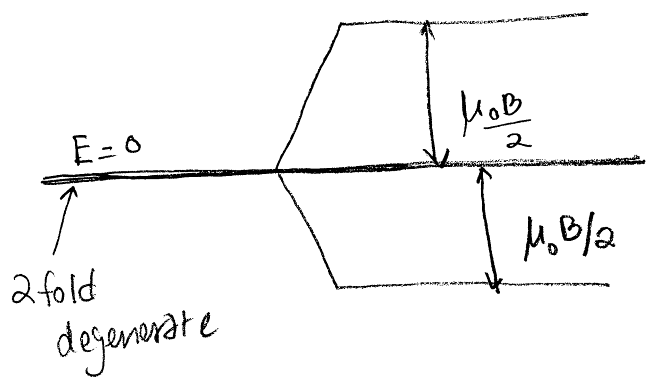
$$\begin{pmatrix} 0 & & & 0 \\ & 0 & & \\ & & 0 & \\ 0 & & & -1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = 0 \Rightarrow a_4 = 0$$

$$|\lambda = \frac{\mu_0 B}{2}\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$|\lambda = -\frac{\mu B_0}{2}\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$|\lambda = 0, 1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$|\lambda = 0, 2\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$



b) $$H' = -\bar{\mu}_2 \cdot \bar{B}_{21} = \underbrace{\frac{g^2 \mu_0^2}{\hbar^2 d^3}}_{\alpha} \left(-3 S_z(1) S_z(2) + \bar{S}(1) \cdot \bar{S}(2)\right)$$

$$= \alpha \left\{ -3 S_{1z} S_{2z} + S_{1x} S_{2x} + S_{1y} S_{2y} + S_{1z} S_{2z} \right\}$$

$$\left.\begin{aligned} S_+ &= S_x + i S_y \\ S_- &= S_x - i S_y \end{aligned}\right\} \begin{aligned} &\Rightarrow S_x = \frac{1}{2}(S_+ + S_-) \\ &\Rightarrow S_y = \frac{1}{2i}(S_+ - S_-) \end{aligned}$$

$$H' = \alpha \left\{ -2 S_{1z} S_{2z} + \frac{1}{4}(S_{1+} + S_{1-})(S_{2+} + S_{2-}) - \frac{1}{4}(S_{1+} - S_{1-})(S_{2+} - S_{2-}) \right\}$$

$$= \alpha \left\{ -2 S_{1z} S_{2z} + \frac{1}{4} \left\{ S_{1+} S_{2+} + S_{1+} S_{2-} + S_{1-} S_{2+} + S_{1-} S_{2-} - S_{1+} S_{2+} + S_{1+} S_{2-} + S_{1-} S_{2+} - S_{1-} S_{2-} \right\} \right.$$

$$= \alpha \left\{ -2 S_{1z} S_{2z} + \frac{1}{2}(S_{1+} S_{2-} + S_{1-} S_{2+}) \right\}$$

$$H'|1\rangle = \alpha \left\{ -2 S_{1z} S_{2z} + \frac{1}{2}(S_{1+} S_{2-} + S_{1-} S_{2+}) \right\} |++\rangle$$

$$= \alpha \left\{ -\frac{\hbar^2}{2} \right\} |++\rangle$$

$$= -\frac{g^2 \mu_0^2}{2 d^3} |++\rangle$$

$$H'|2\rangle = \alpha \left\{ -2 S_{1z} S_{2z} + \frac{1}{2}(S_{1+} S_{2-} + S_{1-} S_{2+}) \right\} |+-\rangle$$

$$= \alpha \left\{ \frac{\hbar^2}{2} |+-\rangle + \frac{\hbar^2}{2} \sqrt{\frac{3}{4} - \frac{1}{2}\left(\frac{1}{2} - 1\right)} \sqrt{\frac{3}{4} - \left(-\frac{1}{2}\right)\left(-\frac{1}{2} + 1\right)} |-+\rangle \right\}$$

$$= \alpha \left\{ \frac{\hbar^2}{2} |+-\rangle + \frac{\hbar^2}{2} |-+\rangle \right\}$$

$$H'|3\rangle = \alpha\left\{ -2S_{1z}S_{2z} + \frac{1}{2}\left(S_{1+}S_{2-} + S_{1-}S_{2+}\right)\right\} |-+\rangle$$

$$= 4\alpha\left\{ \frac{\hbar^2}{2}|-+\rangle + \frac{\hbar^2}{2}\sqrt{\frac{3}{4} - \left(-\frac{1}{2}\right)\left(-\frac{1}{2}+1\right)}\sqrt{\frac{3}{4} - \frac{1}{2}\left(\frac{1}{2}-1\right)} \; |+-\rangle \right.$$

$$= \alpha\left\{ \frac{\hbar^2}{2}|-+\rangle + \frac{\hbar^2}{2}|+-\rangle \right\}$$

$$H'|4\rangle = \alpha\left\{ -2S_{1z}S_{2z} + \frac{1}{2}\left(S_{1+}S_{2-} + S_{1-}S_{2+}\right)\right\} |--\rangle$$

$$= \alpha\left\{ -\frac{\hbar^2}{2}|--\rangle \right\}$$

$$H' = \begin{array}{cc} & \begin{array}{cccc} |1\rangle & |2\rangle & |3\rangle & |4\rangle \\ |++\rangle & |+-\rangle & |-+\rangle & |--\rangle \end{array} \\ \begin{array}{c} \langle 1| \; \langle ++| \\ \langle 2| \; \langle +-| \\ \langle 3| \; \langle -+| \\ \langle 4| \; \langle --| \end{array} & \begin{pmatrix} -\alpha & 0 & 0 & 0 \\ 0 & 2\alpha & 2\alpha & 0 \\ 0 & 2\alpha & 2\alpha & 0 \\ 0 & 0 & 0 & -\alpha \end{pmatrix} \frac{\hbar^2}{2} \end{array}$$

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PROBLEM 4

Consider a two state system described by the time-dependent Hamiltonian

$$H = \begin{pmatrix} 0 & \frac{\beta}{2}e^{i\omega t} \\ \frac{\beta}{2}^* e^{-i\omega t} & \hbar\omega_1 \end{pmatrix}$$

with

$$\vec{v}(t) = \begin{pmatrix} v_o(t) \\ v_1(t) \end{pmatrix}.$$

This is the Hamiltonian of a spin 1/2 particle in a strong magnetic field in the $\hat{z}$ direction combined with a rotating magnetic field in the x-y plane and models many NMR experiments. To analyze this system, it is convenient to write $\vec{v}(t)$ in terms of the time dependent vector $\vec{s}(t) = \begin{pmatrix} s_o(t) \\ s_1(t) \end{pmatrix}$ so that

$$\vec{v}(t) = \begin{pmatrix} s_o(t) \\ s_1(t)e^{-i\omega t} \end{pmatrix}.$$

For the case that $\beta = 0$ and $\omega = \omega_1$ (no rotating magnetic field), $s_o(t)$ and $s_1(t)$ are constant. The time dependence of $s_o(t)$ and $s_1(t)$ allows us to determine the probability that the rotating magnetic field induces a spin flip.

a) [1pt] Show that for $\beta = 0$, $\vec{v}(t)$ statisfies the time-dependent Schrodinger equation

$$H\vec{v}(t) = i\hbar\frac{\partial\vec{v}(t)}{\partial t}.$$

when when $s_o(t)$ and $s_1(t)$ are constant and $\omega = \omega_1$.

b) [3pts] For the case that β is a nonzero constant, use Schrödinger's equation for $\vec{v}(t)$ to show that $\vec{s}(t)$ evolves according to the effective Hamiltonian H' with

$$H' = \begin{pmatrix} 0 & \frac{\beta}{2} \\ \frac{\beta}{2}^* & \hbar\Delta\omega \end{pmatrix}$$

and

$$\Delta\omega = \omega_1 - \omega.$$

c) [3pts]Assuming the system starts in the state $\vec{s}(t) = \binom{0}{1}$ at $t = 0$, find $\vec{s}(t)$.

d) [3pts]Assuming the system starts in the state $\vec{s}(t) = \binom{0}{1}$ at $t = 0$, find the probability of finding the system in the state $\binom{1}{0}$ as a function of time.

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PROBLEM 5

A particle of mass m is confined to a two-dimensional plane. The potential energy of the particle is

$$V(\rho) = \begin{cases} 0 & \rho < \rho_o \\ \infty & \rho \geq \rho_o, \end{cases}$$

where ρ is the radial coordinate of plane polar coordinate (ρ, φ). This potential confines the particle to the region of space $\rho \leq \rho_o$. The particle in this "circular square well" is the quantum analog of a marble on the head of a drum. The stationary-state Hamiltonian eigenfunctions of the particle are $\Psi_{n,m_\ell}(\rho, \varphi)$ with eigenenergies E.

a) [4pts] Write down a second-order differential equation for the radial function $R_{n,m_\ell}(\rho)$ in the bound-state Hamiltonian eigen functions

$$\psi_{n,m_\ell}(\rho, \varphi) = R_{n,m_\ell}(\rho)\Phi_{m_\ell}(\varphi),$$

where $\Phi_{m_\ell}(\varphi)$ is an eigenfunction of the orbital angular momentum operator $\hat{L} = -i\hbar\partial/\partial\varphi$. Write down and justify the boundary conditions that physically admissible solutions to your differential equation must satisfy, and write down the normalization integral for the radial functions.

b) [2pts] What, if anything, can you conclude from your differential equation about the degree of degeneracy of the bound-state energies E_{n,m_ℓ}. Justify your answer.

c) [2pts] Derive an equation for the bound-state energies E_{n,m_ℓ} in terms of the zeros $\varsigma_{n,\nu}$ of the cylindrical Bessel function of the first kind, $J_{\pm\nu}(z)$. (See the hint below.)

d) [2pts] Estimate the energies of the *lowest three* bound states of the cylindrical square well. Express your answers in terms of fundamental constants, the mass m, and the well radius ρ_o.

Hint: The cylindrical Bessel functions are solutions of Bessel's equation

$$\left[z^2\frac{d^2}{dz^2} + z\frac{d}{dz} + (z^2 - \nu^2)\right] J_{\pm\nu}(z) = 0$$

The so-called cylindrical Bessel functions of the first kind, $J_{\pm\nu}(z)$, are regular at the origin and normalizable. These functions oscillate with increasing z and have an infinite number of *nodes*, i.e., values for which $z = \varsigma_{n,\nu} > 0$ at which $J_{\pm\nu}(z) = 0$; these nodes are indexed by $n = 1, 2, ...$ The figure shows the first four cylindrical Bessel functions.

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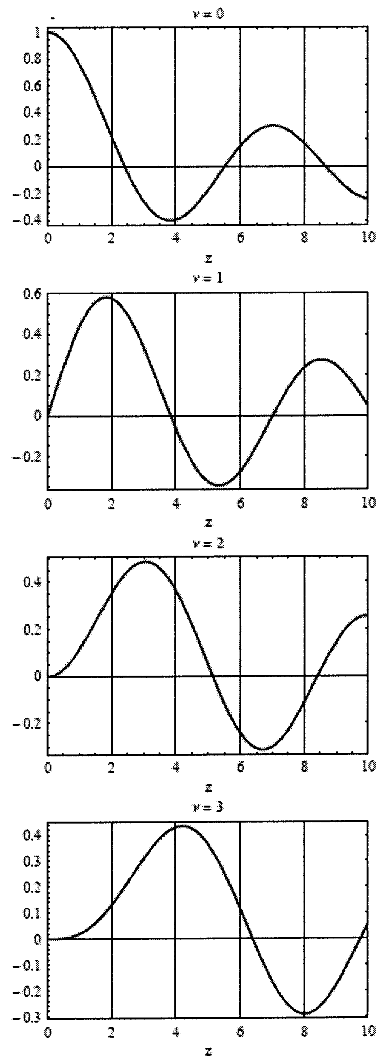

First four cylindrical Bessel functions of the first kind (for use in problem 5.)

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PROBLEM 6

Consider an ensemble of identical particles whose state space is spanned by the basis

$$|e_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |e_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |e_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Assume that the Hamiltonian H and an observable A are represented by

$$H = \hbar\omega_o \begin{pmatrix} 0 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \text{ and } A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The eigenvalues of H are $\hbar\omega$, $2\hbar\omega$, and $-\hbar\omega$ with eigenvectors given by

$$|\hbar\omega\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad |2\hbar\omega\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \text{ and } |-\hbar\omega\rangle \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

The eigenvalues of A are $-1, 1,$and 1 with eigenvectors given by

$$|a_{-1}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad |a_{1,1}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad |a_{1,2}\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

For all times $t < 0$, the particles are in a state given by

$$|\psi_o\rangle = \frac{1}{2} \begin{pmatrix} 1 \\ \sqrt{2} \\ -1 \end{pmatrix}.$$

a) [1pt] Write down an expression for the time evolution operator $U(t, t_o = 0)$ in Dirac notation

b) [2 pt] Determine $|\psi(t)\rangle$, the state vector at an arbitrary time.

c) [2 pt] What is the probability that a measurement of A at a time $t = 0$ yeilds $a = -1$?

d) [2 pt] What is the probability that a measurement of A at an arbitrary time t yields a value $a = -1$?

e) [3 pt] Assume that at $t = 0$ the operator A is observed to be 1. What is the probability that a short time later ($0 < t << 1/\omega$), the eigenenergy of the system is observed to be $-\hbar\omega$?

P6.

a) $$U(t,t_0=0) = \sum_{E_i} |E_i\rangle\langle E_i| \, e^{-iE_i(t-t_0)/\hbar}$$

$$= |E=2\hbar\omega\rangle\langle E=2\hbar\omega| \, e^{-i2\omega t} + |E=\hbar\omega\rangle\langle E=\hbar\omega| \, e^{-i\omega t}$$

$$+ |E=-\hbar\omega\rangle\langle E=-\hbar\omega| \, e^{i\omega t}$$

b) $$|\psi(0)\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ -1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/\sqrt{2} \\ -1/2 \end{pmatrix} = \frac{1}{\sqrt{2}} |E=\hbar\omega\rangle + \frac{1}{\sqrt{2}} |E=\hbar\omega\rangle$$

$$|\psi(t)\rangle = U(t,0)\,|\psi(0)\rangle = \frac{e^{-i\omega t}}{\sqrt{2}} |E=\hbar\omega\rangle + \frac{e^{-i2\omega t}}{\sqrt{2}} |E=2\hbar\omega\rangle$$

c) $$|\psi(0)\rangle = \begin{pmatrix} 1/2 \\ 1/\sqrt{2} \\ -1/2 \end{pmatrix} = \frac{1}{2}|e_1\rangle + \frac{1}{\sqrt{2}}|e_2\rangle - \frac{1}{2}|e_3\rangle$$

$$|a=-1\rangle = \frac{1}{\sqrt{2}}\left[|e_1\rangle - |e_2\rangle\right]$$

$$|a=1,1\rangle = \frac{1}{\sqrt{2}}\left[|e_1\rangle + |e_2\rangle\right]$$

$$|a=1,2\rangle = |e_3\rangle$$

$$|\psi(0)\rangle_{|a\rangle} = \{|a=-1\rangle\langle a=-1| + |a=1,1\rangle\langle a=1,1| + |a=1,2\rangle\langle a=1,2|\}$$

$$|\psi(0)\rangle$$

$$P(a=-1) = \frac{|\langle a=-1|\psi(0)\rangle|^2}{\langle\psi(0)|\psi(0)\rangle}$$

$$\langle\psi(0)|\psi(0)\rangle = \left(\tfrac{1}{2} \ \tfrac{1}{\sqrt{2}} \ -\tfrac{1}{2}\right)\begin{pmatrix}\frac{1}{2}\\ \frac{1}{\sqrt{2}}\\ -\frac{1}{2}\end{pmatrix} = \frac{1}{4}+\frac{1}{2}+\frac{1}{4} = 1$$

$$\langle a=-1|\psi(0)\rangle = \left(\tfrac{1}{\sqrt{2}} \ -\tfrac{1}{\sqrt{2}} \ 0\right)\begin{pmatrix}\frac{1}{2}\\ \frac{1}{\sqrt{2}}\\ -\frac{1}{2}\end{pmatrix} = \left(\frac{1}{2\sqrt{2}} - \frac{1}{2}\right)$$

$$P(a=1) = \frac{1}{4}\left(\frac{1}{\sqrt{2}}-1\right)^2 = \frac{(1-\sqrt{2})^2}{8} = \frac{3-2\sqrt{2}}{8} \quad ??$$

$$|\psi(t)\rangle = \frac{e^{-i\omega t}}{2}\begin{pmatrix}1\\0\\-1\end{pmatrix} + \frac{e^{-i2\omega t}}{\sqrt{2}}\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}e^{-i\omega t}/2\\ e^{-i2\omega t}/\sqrt{2}\\ -e^{-i\omega t}/2\end{pmatrix}$$

$$\langle a=-1|\psi(t)\rangle = \left(\tfrac{1}{\sqrt{2}} \ -\tfrac{1}{\sqrt{2}} \ 0\right)\begin{pmatrix}\frac{1}{2}e^{-i\omega t}\\ \frac{1}{\sqrt{2}}e^{-i2\omega t}\\ -\frac{1}{2}e^{-i\omega t}\end{pmatrix} = \frac{1}{2\sqrt{2}}e^{-i\omega t} - \frac{1}{2}e^{-i2\omega t}$$

$$\langle\psi(t)|\psi(t)\rangle = \left(\tfrac{1}{2}e^{+i\omega t} \ \ \tfrac{1}{\sqrt{2}}e^{+i2\omega t} \ \ -\tfrac{1}{2}e^{+i\omega t}\right)\begin{pmatrix}\frac{1}{2}e^{-i\omega t}\\ \frac{1}{\sqrt{2}}e^{-i2\omega t}\\ -\frac{1}{2}e^{-i\omega t}\end{pmatrix}$$

$$= \frac{1}{4}+\frac{1}{2}+\frac{1}{4} = 1$$

$$P(t)\,(a=-1) = \left|\frac{1}{2\sqrt{2}}\,e^{-i\omega t} - \frac{1}{2}\,e^{-i2\omega t}\right|^2$$

$$= \left(\frac{1}{2\sqrt{2}}\,e^{-i\omega t} - \frac{1}{2}\,e^{-i2\omega t}\right)\left(\frac{1}{2\sqrt{2}}\,e^{i\omega t} - \frac{1}{2}\,e^{i2\omega t}\right)$$

$$= \frac{1}{8} - \frac{1}{4\sqrt{2}}\,e^{i\omega t} - \frac{1}{4\sqrt{2}}\,e^{-i\omega t} + \frac{1}{4}$$

$$= \frac{3}{8} - \frac{1}{2\sqrt{2}}\cos\omega t$$

$$\frac{1}{2\sqrt{2}}\,(1\ -1\ \ 0)\begin{pmatrix}1\\ \sqrt{2}\\ -1\end{pmatrix}|a=-1\rangle + \frac{1}{2\sqrt{2}}\,(1\ \ 1\ \ 0)\begin{pmatrix}1\\ \sqrt{2}\\ -1\end{pmatrix}|a=1,1\rangle$$

$$+\ \frac{1}{2}\,(0\ \ 0\ \ 1)\begin{pmatrix}1\\ \sqrt{2}\\ -1\end{pmatrix}|a=1,2\rangle$$

$$|\psi(0)\rangle_{|a\rangle} = \frac{1}{2\sqrt{2}}(1-\sqrt{2})\,|a=-1\rangle + \frac{1}{2\sqrt{2}}(1+\sqrt{2})\,|a=1,1\rangle - \frac{1}{2}\,|a=1,2\rangle$$

$$P(a=-1) = \frac{(1-\sqrt{2})^2}{8}$$

since $P(A) = 1$

e) $|\psi(0)\rangle = |a=-1\rangle + |a=1,1\rangle + |a=1,2\rangle$

$=$