

Quantum Mechanics

Qualifying Exam—January 2006

Notes and Instructions:

- There are **6** problems and **7** pages.
- Be sure to write your alias at the top of every page.
- Number each page with the problem number, and page number of your solution (e.g. “Problem 3, p. 1/4” is the first page of a four page solution to problem 3).
- **You must show all your work.**

Possibly useful formulas:

Pauli spin matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

One-dimensional simple harmonic oscillator operators:

$$X = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger)$$

$$P = -i\sqrt{\frac{\hbar m\omega}{2}}(a - a^\dagger)$$

Spherical Harmonics:

$$Y_0^0(\theta, \varphi) = \frac{1}{\sqrt{4\pi}} \quad Y_2^2(\theta, \varphi) = \frac{5}{\sqrt{96\pi}} 3 \sin^2 \theta e^{2i\varphi}$$

$$Y_2^1(\theta, \varphi) = -\frac{5}{\sqrt{24\pi}} 3 \sin \theta \cos \theta e^{i\varphi}$$

$$Y_1^1(\theta, \varphi) = -\frac{3}{\sqrt{8\pi}} \sin \theta e^{i\varphi} \quad Y_2^0(\theta, \varphi) = \frac{5}{\sqrt{4\pi}} \left(\frac{3}{2} \cos^2 \theta - \frac{1}{2} \right)$$

$$Y_1^0(\theta, \varphi) = \frac{3}{\sqrt{4\pi}} \cos \theta \quad Y_2^{-1}(\theta, \varphi) = \frac{5}{\sqrt{24\pi}} 3 \sin \theta \cos \theta e^{-i\varphi}$$

$$Y_1^{-1}(\theta, \varphi) = \frac{3}{\sqrt{8\pi}} \sin \theta e^{-i\varphi} \quad Y_2^{-2}(\theta, \varphi) = \frac{5}{\sqrt{96\pi}} 3 \sin^2 \theta e^{-2i\varphi}$$

Angular momentum raising and lowering operators:

$$\hat{L}_\pm = (\hat{L}_x \pm i \hat{L}_y)$$

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PROBLEM 1: Eigenvalues and Eigenvectors

Suppose the Hamiltonian for a system is given by

$$H = \hbar\omega_0(\sigma_x + \sigma_y)$$

where σ_x and σ_y are two of the Pauli matrices.

- (a). Calculate the eigenvalues and eigenvectors for this Hamiltonian. (5 points)
- (b). In the Schrödinger picture, the state vector is

$$|\psi(t)\rangle = \begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix}$$

At $t = 0$, the state of this system has $\alpha(0) = 0$ and $\beta(0) = 1$. Evaluate $\alpha(t)$ and $\beta(t)$ for $t > 0$. (5 points)

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PROBLEM 2: One Dimensional Barrier

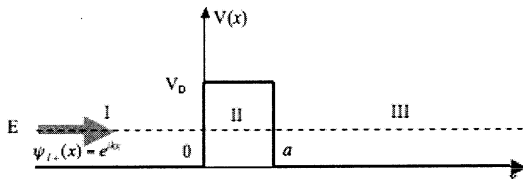


Figure 1: A one-dimensional rectangular barrier.

A stream of particles with kinetic energy $0 < E < V_0$ and mass m , traveling in the positive x direction is incident on a barrier from the left. These incident particles have a wave function

$$\psi_{in}(x) = e^{ikx}$$

where $k = \sqrt{2mE}/\hbar$.

- Write down the wave function in each region, ($\psi_I(x)$, $\psi_{II}(x)$, and $\psi_{III}(x)$), in terms of the eigenfunctions in each region, with undetermined coefficients. (2 points)
- State the boundary conditions at $x = 0$, $x = a$, and $x = \infty$. (1 point)
- Sketch the wavefunction in a single figure, showing its form in all regions. (1 point)
- Find the transmission coefficient (T) for particles with

$$k = \sqrt{2mE}/\hbar$$

$$\lambda = \sqrt{2m(V_0 - E)}/\hbar.$$

(4 points)

- If the barrier of transmission is small ($\lambda a \ll 1$), show that

$$T \approx \frac{16E}{V_0} \left(1 - \frac{E}{V_0}\right) \exp\left(-\frac{2a}{\hbar} \sqrt{2m(V_0 - E)}\right).$$

(2 points)

PROBLEM 3: Harmonic Oscillator in 3 Dimensions

Consider a particle subject to a 3-dimensional harmonic oscillator potential:

$$\begin{aligned} H &= H_x + H_y + H_z \\ &= \frac{P_x^2}{2m} + \frac{1}{2}m\omega_x^2 X^2 + \frac{P_y^2}{2m} + \frac{1}{2}m\omega_y^2 Y^2 + \frac{P_z^2}{2m} + \frac{1}{2}m\omega_z^2 Z^2 \end{aligned}$$

where $\omega_x = \omega_y = \omega$ and $\omega_z = 2\omega$. The wave function is given by:

$$\Psi_{n_x, n_y, n_z}(x, y, z) = \Phi_{n_x}(x)\Phi_{n_y}(y)\Phi_{n_z}(z)$$

- (a). For the ground state wave function, determine, ΔX , ΔP_x and their product $\Delta X \Delta P_x$ using Dirac notation and raising and lowering operators. (3 points)
- (b). For the ground state, $\Psi_{0,0,0}$, do you expect
- ΔZ to be larger or smaller than ΔX ?
 - $\Delta Z \Delta P_z$ to be larger or smaller than $\Delta X \Delta P_x$?
- Explain your reasoning in each case. (2 points)
- (c). Assume that at $t = 0$ the particle is in the state:

$$|\Psi(t=0)\rangle = \frac{1}{\sqrt{6}}|\Psi_{1,0,0}\rangle + \frac{1}{\sqrt{3}}|\Psi_{2,1,0}\rangle + \frac{1}{2}|\Psi_{0,1,0}\rangle + \frac{1}{2}|\Psi_{1,0,1}\rangle$$

If one measures the total energy, E , what is the probability of obtaining $5\hbar\omega$? (2 points)

- (d). Immediately after the measurement performed in part (c), what harmonic oscillator state (or superposition of states) is the system in? (1 points)
- (e). Assume that at time $t = 0$ the measurement described in part (c) is performed and that the energy is found to be $E = 5\hbar\omega$. If the observable, X , is measured at a time $t > 0$, will its probability distribution be time dependent or time independent? Explain your reasoning. (2 points)

$$\frac{P_3}{a)} \quad (\Delta X) = \sqrt{\langle X^2 \rangle - \langle X \rangle^2}$$

$$\langle X \rangle = \langle \Psi_{000} | x | \Psi_{000} \rangle = \sqrt{\frac{\hbar}{2m\omega}} \langle \Psi_{000} | a + a^\dagger | \Psi_{000} \rangle = 0$$

$$\langle X^2 \rangle = \frac{\hbar}{2m\omega} \langle \Psi_{000} | (a^\dagger)^2 + (a)^2 + aa^\dagger + a^\dagger a | \Psi_{000} \rangle = (1) \frac{\hbar}{2m\omega}$$

$$\Delta X = \sqrt{\frac{\hbar}{2m\omega}}$$

$$\Delta P_x = \sqrt{\langle P_x^2 \rangle - \langle P_x \rangle^2}$$

$$\langle P_x \rangle = -i\sqrt{\frac{\hbar m\omega}{2}} \langle \Psi_{000} | a - a^\dagger | \Psi_{000} \rangle = 0$$

$$\langle P_x^2 \rangle = -\frac{\hbar m\omega}{2} \langle \Psi_{000} | (a)^2 + (a^\dagger)^2 - aa^\dagger - a^\dagger a | \Psi_{000} \rangle = +\frac{\hbar m\omega}{2}$$

$$\Delta P_x = \sqrt{\frac{\hbar m\omega}{2}}$$

$$\Delta X \Delta P_x = \frac{\hbar}{2}$$

b) i) smaller ii) same they are decoupled

$$c) \quad H|\Psi_{100}\rangle = \hbar\omega_x(1+\frac{1}{2}) + \hbar\omega_y(0+\frac{1}{2}) + \hbar\omega_z(0+\frac{1}{2})|\rangle = \hbar\omega(\frac{3}{2} + \frac{1}{2} + \frac{1}{2})|\rangle = 3\hbar\omega$$

$$\rightarrow H|\Psi_{210}\rangle = \hbar\omega_x(2+\frac{1}{2}) + \hbar\omega_y(1+\frac{1}{2}) + \hbar\omega_z(0+\frac{1}{2})|\rangle = \hbar\omega(\frac{5}{2} + \frac{3}{2} + \frac{1}{2})|\rangle = 5\hbar\omega|\rangle$$

$$H|\Psi_{010}\rangle = (\frac{\hbar\omega}{2} + \frac{3\hbar\omega}{2} + \hbar\omega)|\rangle = 3\hbar\omega$$

$$\rightarrow H|\Psi_{1001}\rangle = (\frac{3\hbar\omega}{2} + \frac{\hbar\omega}{2} + 3\hbar\omega) = 5\hbar\omega$$

$$P(E = 5\hbar\omega) = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$$

$$\rightarrow d) |\psi(t=0)\rangle = (\frac{1}{\sqrt{3}}|\Psi_{210}\rangle + \frac{1}{2}|\Psi_{101}\rangle) \times \sqrt{\frac{12}{7}}$$

PROBLEM 4: Angular Momentum

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2+211

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Consider the problem of a 3D rigid rotator. Let the rotator be far removed from any forces so its energy is purely kinetic.

$$E = \frac{L^2}{2I}$$

where I is the moment of inertia. The quantum mechanical Hamiltonian operator is

$$H = \frac{\hat{L}^2}{2I}$$

↖ 220

- (a). What are the frequencies of photons emitted due to the energy decay between successive levels of the rotator? (3 points)

At a given instant, the rigid rotator is in the state

$$\Psi(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \sin \theta \sin \phi$$

- (b). What possible values of L_z will be found in a measurement, and with what probabilities? (3 points)
- (c). What is $\langle L_x \rangle$ for this state? (2 points)
- (d). What is $\langle L^2 \rangle$ for this state? (2 points)

$$a) \quad \hat{H} \Psi(\theta, \phi) = E \Psi(\theta, \phi)$$

$$\Rightarrow \frac{\hat{L}^2}{2I} \Psi(\theta, \phi) = E \Psi(\theta, \phi)$$

$$\hat{L}^2 = -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right]$$

$$\Rightarrow \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} (\sin\phi) = -\frac{\cos\phi}{\sin\theta}$$

$$\begin{aligned} \Rightarrow \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial \sin\theta}{\partial\theta} \right) &= \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} (\sin\theta \cos\theta) \\ &= \frac{1}{\sin\theta} (\cos^2\theta - \sin^2\theta) = \frac{\cos^2\theta - \sin^2\theta}{\sin\theta} \end{aligned}$$

thus,

$$\hat{H} \Psi = E \Psi$$

$$\Rightarrow \frac{-\hbar^2}{2I} \left[\sqrt{\frac{3}{4\pi}} \sin\phi \frac{\cos^2\theta - \sin^2\theta}{\sin\theta} - \sqrt{\frac{3}{4}} \cos\phi \right] = E \Psi$$

$$\Rightarrow \underbrace{\sqrt{\frac{3}{4\pi}} \sin\theta \sin\phi}_{\Psi} \left\{ -\frac{\hbar^2}{2I} \left[\frac{\cos^2\theta - \sin^2\theta}{\sin^2\theta} - \frac{\cot\phi}{\sin\theta} \right] \right\} = E \Psi$$

$$\frac{-\hbar^2}{2I} \left[\frac{\cos^2 \theta - \sin^2 \theta - \cot \phi \sin \theta}{\sin^2 \theta} \right] = E$$

$$\Rightarrow \omega = -\frac{\hbar}{2I} \left[\cot^2 \theta - 1 - \frac{\cot \phi}{\sin \theta} \right]$$

this is bullshit
 what I did nevermind.

$$b) \hat{L}_z = -i\hbar \frac{\partial}{\partial \phi}$$

$$\hat{L}_z \Psi(\theta, \phi) = -i\hbar \cos\theta \sqrt{\frac{3}{4\pi}} \sin\theta$$

$$= -i\hbar \cot\theta \sqrt{\frac{3}{4\pi}} \sin\theta \sin\phi = m\hbar$$

$$\Rightarrow m = -i \cot\theta$$

$$e^{-im\phi} = \sin\phi$$

$$\Rightarrow \cos m\phi - i \sin m\phi = \sin\phi$$

PROBLEM 5: Variational Method

Let us consider an electron moving in a Coulomb potential:

$$H = T + V = -\frac{\hbar^2}{2m}\nabla^2 - \frac{e^2}{r}$$

where m is the mass of the electron.

Let us choose

$$\psi_\alpha(r) = e^{-\alpha r^2}, \quad \alpha > 0$$

as a trial wave function for the ground state.

- (a). Find $\langle \psi_\alpha | \psi_\alpha \rangle$. (1 points)
- (b). Find the expectation value of the Hamiltonian $\langle H \rangle$. (3 points)
- (c). Determine a bound on the ground state energy of this system using the variational method and this trial wavefunction. Express your answer in terms of

$$\text{Ry} = \frac{me^4}{2\hbar^2}.$$

(5 points)

- (d). How does this compare to the actual ground state energy of the system? Be specific. (1 point)

PROBLEM 6: Perturbation Theory

Consider a particle confined to a ring of unit radius. You are told that the Hamiltonian operator can be written as $H = H_0 + H_1$ where

$$H_0 = \left(i \frac{\partial}{\partial \theta} + a\right)^2$$

and

$$H_1 = V_0 \cos \theta .$$

The parameter a is a tunable constant in this toy model. (In this problem, units are chosen such that $\hbar = 2m = 1$)

- (a). Find the complete set of eigenvalues and eigenfunctions of H_0 (2 points)
- (b). Use perturbation theory to find the first order correction to the ground state energy of H_0 due to the perturbation H' for $0 < a < 1/2$. (2 points)
- (c). Use perturbation theory to find the second order corrections to the ground state energy of H_0 due to the perturbation H_1 for $0 < a < 1/2$. (3 points)
- (d). For $a = 1/2$, the ground state energy of H_0 is degenerate. Find the first order correction to the energy for this case. (3 points)

