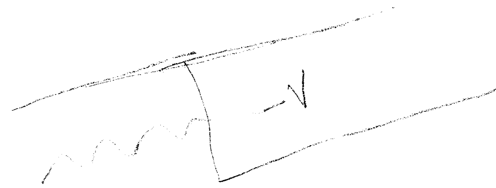
Aug 2009

F-9

Problem 1: Step Potential (10 points)

Consider the potential $V(x)$



$$V(x) = \begin{cases} 0, & x \leq 0 \\ -\mathrm{V}, & x > 0 \end{cases}$$

A particle of mass m and kinetic energy E approaches the step from $x < 0$.

a) Write the solution to Schrodinger's equation for $x < 0$. (1 pt)

b) Write the solution to Schrodinger's equation for $x > 0$. (1 pt)

c) Sketch the wave function for $x < 0$ as well as $x > 0$. Making sure to describe how the amplitude and frequency of the wave function changes. (1 pt)

d) What is the probability that particle will reflect back if $E = V/8$? (2 pts)

e) What is the probability that the particle will be transmitted if $E = V/8$. (2 pts) (Determine the transmission probability directly by using the flow of probability current and do not simply use $T = 1 - R$)

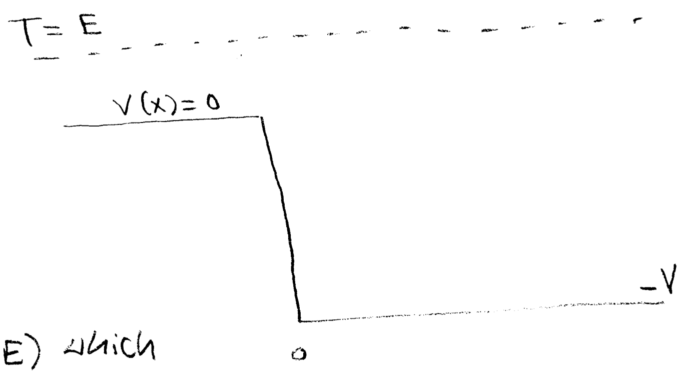
f) Show that $T + R = 1$. What does this mean physically? (1 pt)

g) If instead the particle approached the step from $x > 0$, how do your answers to parts a), b), d) and e) change? (2 pts)

I have to do it again (part g)

F-09

* total energy, E = E



P1. $V(x) = \begin{cases} 0 \; ; & x \leq 0 \\ -V \; ; & x > 0 \end{cases}$

Notice, the total energy is E ($\because T = E$) which is pos.

a) $$\frac{d^2\psi_I(x)}{dx^2} + \frac{2mE}{\hbar^2}\psi_I(x) = 0$$

$$\Rightarrow \frac{d^2\psi_I(x)}{dx^2} = -k^2\psi_I(x) = 0 \qquad k^2 = \frac{2mE}{\hbar^2}$$

$$\Rightarrow \psi_I(x) = Ae^{ikx} + Be^{-ikx}$$

b) $$\frac{d^2\psi_{II}(x)}{dx^2} + \frac{2mV}{\hbar^2}\psi_{II}(x) = -\frac{2mE}{\hbar^2}\psi_{II}(x) \qquad \frac{2mT}{\hbar^2}$$

$$\Rightarrow \frac{d^2\psi_{II}(x)}{dx^2} = -\underbrace{\frac{2m(V+E)}{\hbar^2}}_{\alpha^2}\psi_{II}(x)$$

$$\Rightarrow \lambda = \pm i\alpha$$

$$\psi_{II}(x) = Ce^{i\alpha x} + \cancel{De^{-i\alpha x}}^{\,0}$$ as the particle is coming from $-\infty$

$$= Ce^{i\alpha x}$$

Matching the wave fon^c

$$\psi_I(0) = \psi_{II}(0)$$

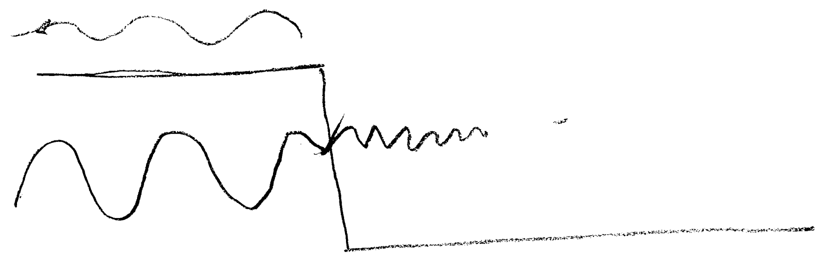
$$\Rightarrow A + B = C \quad (1)$$

$$\left.\frac{d\psi_I}{dx}\right|_{x=0} = \left.\frac{d\psi_{II}}{dx}\right|_{x=0}$$

$$\Rightarrow ik(Ae^{+ikx} - Be^{-ikx})\Big|_{x=0} = i\alpha Ce^{i\alpha x}\Big|_{x=0}$$

$$\Rightarrow k(A-B) = \alpha C \quad (2)$$

c)



the amplitude & frequency of the reflected wave is same as incident

second thoughts

$T_1 = E \qquad T_2 = E + V \qquad T_2 > T_1 \Rightarrow \frac{1}{|A_2|} > \frac{1}{|A_1|} \Rightarrow A_2 < A_1$

~~and~~ So, the amplitude decreases and frequency increases for transmitted wave.

d) $R = \left|\frac{B}{A}\right|^2$

(1) & (2)

$K(A-B) = \alpha(A+B)$

$\Rightarrow A\left(1-\frac{\alpha}{K}\right) = B\left(1+\frac{\alpha}{K}\right)$

$\Rightarrow \frac{B}{A} = \frac{1-\alpha/K}{1+\alpha/K}$

$R = \frac{(1-\alpha/K)^2}{(1+\alpha/K)^2} = \frac{(1-3)^2}{(1+3)^2}$

$= \frac{4}{16} = \frac{1}{4}$

$T + V(+) = T - V$

$V + E = V + T - V = T$

$\frac{\alpha^2}{K^2} = \frac{2m(V+E)}{\hbar^2} \frac{\hbar^2}{2mE}$

$= \frac{V+E}{E} = \frac{V}{E} + 1$

$= 9$

$\frac{\alpha}{K} = 3$

d) $$J = \frac{\hbar}{2im}\left[\psi^* \frac{d\psi}{dx} - \psi \frac{d\psi^*}{dx}\right]$$

$$T = \cancel{\left|\frac{J_{trans}}{J_{inc}}\right|^2} \qquad T = \frac{J_{trans}}{J_{inc}}$$

$$J_{trans} = \frac{\hbar}{2im}\left[C^2 e^{-i\alpha x}\left(i\alpha e^{i\alpha x}\right) - C^2 e^{i\alpha x}(-i\alpha)e^{-i\alpha x}\right]$$

$$= \frac{\hbar C^2}{2im}(2i\alpha) = \frac{\hbar C^2 \alpha}{m}$$

$$J_{inc} = \frac{\hbar}{2im}\Big[\left(A e^{-ikx} + B e^{ikx}\right)\left\{ikA e^{ikx} - ikB e^{-ikx}\right\}$$
$$- \left(A e^{ikx} + B e^{-ikx}\right)\left\{(-ik)A e^{-ikx} + ikB e^{ikx}\right\}\Big]$$

$$= \frac{\hbar}{2im}\left[ik|A|^2 + ikBA\, e^{2ikx} - ikAB\right.$$

$$J_{inc} = \frac{\hbar}{2im}\left[A e^{-ikx}(ik)A e^{ikx} - A e^{ikx}(-ik)A e^{-ikx}\right]$$

$$= \frac{\hbar |A|^2 k}{m}$$

$$T = \left|\frac{J_{trans}}{J_{inc}}\right| = \left|\frac{\cancel{\hbar} C^2 \alpha}{\cancel{m}} \times \frac{\cancel{m}}{\cancel{\hbar}|A|^2 k}\right| = \left|\frac{C^2}{A^2}\right|\left|\frac{\alpha}{k}\right|$$

(1) & (2)

$$2A = C\left(1+\frac{\alpha}{k}\right)$$

$$\Rightarrow \frac{C}{A} = \frac{2}{1+\frac{\alpha}{k}}$$

$$\Rightarrow T = \frac{2^2}{\left(1+\frac{\alpha}{k}\right)^2}\left(\frac{\alpha}{k}\right)$$

For $E = V/8$ $\quad \frac{\alpha}{k} = 3$

$$= \frac{2^2}{4^2} \cdot 3$$

$$= \frac{3}{4}$$

#

f)

$$T+R = \frac{3}{4} + \frac{1}{4} = 1$$

Conservation of no. of particles!

g) a) $\psi_I(x) = Ae^{-ikx}$

b) $\psi_{II}(x) = Be^{-ikx} + Ce^{-ikx}$

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Problem 2: Variational Method (10 points)

Let us consider the hydrogen atom without spin. The Hamiltonian is

$$H = \frac{P^2}{2m} - \frac{C}{r} \,. \tag{1}$$

Since the ground state is an S state the wave function must be spherically symmetrical. Suppose you could not solve this problem exactly. Estimate the ground state wave function with a Gaussian:

$$\psi(\vec{r}) = Ne^{-r^2/b^2} \,.$$

a) Compute the normalization constant N so that $\psi(\vec{r})$ is correctly normalized. (2 pts)

b) Evaluate the expectation value of H in this state. (3 pts)

c) Find the best estimate for E_0 by applying the variational method. (4 pts)

d) The true ground state energy is

$$E_0 = -\frac{1}{2}(C^2 m) \,.$$

check

How much does your estimate in (c) differ from the correct answer? (1 pt)

P2

$$\int_{-\infty}^{+\infty} \psi^*(r)\, \psi(r)\, d\vec{r}^{\,3} = 1 = \int_0^\infty \int_0^\pi \int_0^{2\pi} \psi^*(r)\, \psi(r)\, r^2 dr \sin\theta d\theta d\phi$$

$$|N|^2 \int_{-\infty}^{+\infty} \underbrace{e^{-2r^2/b^2}}\, d\vec{r}^{\,3} = 1 \longrightarrow \int_{-\infty}^{+\infty} e^{-\alpha r^2} d^3 r$$

$$\Rightarrow |N|^2 \left(\sqrt{\frac{\pi b^2}{2}}\right)^3 = 1 \qquad = \underbrace{\int_{-\infty}^{+\infty} e^{-\alpha x^2} dx}\, \underbrace{\int_{-\infty}^{+\infty} e^{-\alpha y^2} dy}\, \underbrace{\int_{-\infty}^{+\infty} e^{-\alpha z^2} dz}$$

$$\Rightarrow |N|^2 = \left(\frac{2}{\pi b^2}\right)^{3/2} \qquad = \sqrt{\frac{\pi}{\alpha}} \sqrt{\frac{\pi}{\alpha}} \sqrt{\frac{\pi}{\alpha}}$$

$$\Rightarrow N = \left(\frac{2}{\pi b^2}\right)^{3/4}$$

$$\Rightarrow N = \left(\frac{2}{\pi b^2}\right)^{3/4} \qquad = \left(\frac{\pi}{\alpha}\right)^{3/2}$$

$$\therefore \psi(\vec{r}) = \left(\frac{2}{\pi b^2}\right)^{3/4} e^{-r^2/b^2}$$

$$\langle H \rangle = \int_{-\infty}^{+\infty} \psi^*(r) \left(-\frac{\hbar^2}{2m}\frac{d^2}{dr^2} - \frac{c}{r}\right) \psi(r)\, d^3 r$$

$$= \underline{-\frac{\hbar^2}{2m}\int_{-\infty}^{+\infty} \psi^*(r) \frac{d^2}{dr^2}\psi(r)\, d\vec{r}^{\,3}} - \underbrace{c \int_{-\infty}^{+\infty} \psi^*(r) \frac{1}{r} \psi(r)\, d^3 r}$$

$$\Rightarrow \underbrace{-\frac{\hbar^2}{2m}\left(\frac{2}{\pi b^2}\right)^{3/2}}_{\beta} \int_{-\infty}^{+\infty} e^{-r^2/b^2} \frac{d}{dr}\left(-\frac{1}{b^2} 2r\, e^{-r^2/b^2}\right) dr$$

$$\Rightarrow \beta \int_{-\infty}^{+\infty} e^{-r^2/b^2} \left(-\frac{1}{b^2}\right)\left(2 e^{-r^2/b^2} - \frac{1}{b^2} 2r^2\right) e^{-r^2/b^2} \, dr$$

$$= -\frac{2\beta}{b^2} \int_{-\infty}^{+\infty} e^{-2r^2/b^2} \, dr + \frac{2\beta}{b^4} \int_{-\infty}^{+\infty} r^2 e^{-2r^2/b^2} \, dr \quad *$$

cannot do that

$$= -\frac{2\beta}{b^2}\left(\frac{\pi b^2}{2}\right)^{3/2} + \frac{2\beta}{b^4}\left(-\frac{b^2}{2}\right)^3 \left(\sqrt{\frac{\pi b^2}{2}}\right)^{3/2}$$

$$= \frac{2}{b^2}\frac{\hbar^2}{2m}\left(\frac{2}{\pi b^2}\right)^{3/2}\left(\frac{\pi b^2}{2}\right)^{3/2} - \frac{2}{b^4}\frac{\hbar^2}{2m}\left(\frac{2}{\pi b^2}\right)^{3/2}\left(-\frac{b^2}{2}\right)^3\left(\frac{\pi b^2}{2}\right)^{3/2}$$

WRONG

$$= \frac{\hbar^2}{m}\left(\frac{1}{b^2} + \frac{b^2}{8}\right)$$

$$\langle H \rangle = \frac{\hbar^2}{m}\left(\frac{1}{b^2} + \frac{b^2}{8}\right) + \frac{2\sqrt{2}}{\pi^{3/2} b}$$

$$\Rightarrow -C \int_{-\infty}^{+\infty} \psi^*(r) \frac{1}{r} \psi(r) \, d^3r$$

$$= -C \left(\frac{2}{\pi b^2}\right)^{3/2} \int_{-\infty}^{+\infty} \frac{1}{r} e^{-2r^2/b^2} \, dr$$

$$= -C\left(\frac{2}{\pi b^2}\right)^{3/2} 4\pi \int_0^{\infty} r \, e^{-2r^2/b^2} \, dr$$

$$\frac{2r^2}{b^2} = x$$

$$\Rightarrow \frac{4r\,dr}{b^2} = dx$$

$$\Rightarrow r\,dr = \frac{b^2}{4} dx$$

$$= -C\left(\frac{2}{\pi b^2}\right)^{3/2} \pi b^2 \int_0^{\infty} e^{-x} \, dx$$

$$= -C\left(\frac{2}{\pi b^2}\right)^{3/2} \pi b^2 \left[e^{-x}\right]_0^{\infty} = C\left(\frac{2}{\pi b^2}\right)^{3/2} \pi b^2 = \frac{2\sqrt{2}\,C}{\pi^{1/2} b} \Big\} \text{3rd term}$$

$\Rightarrow$

c) $$\frac{d\langle H\rangle}{db} = \frac{\hbar^2}{m}\left(\frac{-2}{b^3}+\frac{b}{4}\right) - \frac{2\sqrt{2}}{\pi^{3/2}}\frac{1}{b^2} = 0$$

WRONG

$$= b^2\left(-\frac{2}{b^3}+\frac{b}{4}\right) = \frac{m}{\hbar^2}\frac{2\sqrt{2}}{\pi^{3/2}}$$

$$\Rightarrow -\frac{2}{b}+\frac{b^3}{4} = \frac{m}{\hbar^2}\frac{2\sqrt{2}}{\pi^{3/2}}$$

$$* \int_{-\infty}^{+\infty} r^2 e^{-2r^2/b^2}\,dr = \int_{-\infty}^{+\infty} r^2 e^{-ar^2}\,dr$$

$$= -\frac{d}{da}\int_{-\infty}^{+\infty} e^{-ar^2}\,dr = -\frac{d}{da}\left(\sqrt{\frac{\pi}{a}}\right)^{3} = -\frac{3}{2}\left(\frac{\pi}{a}\right)^{+\frac{1}{2}}\left(-\frac{\pi}{a^2}\right)$$

$$= \frac{3}{2a}\left(\frac{\pi}{a}\right)^{3/2}$$

$$= \frac{3b^2}{4}\left(\frac{\pi b^2}{2}\right)^{3/2}$$

$$-\frac{2\beta}{b^2}\left(\frac{\pi b^2}{2}\right)^{3/2} + \frac{3\beta}{2b^2}\left(\frac{\pi b^2}{2}\right)^{3/2}$$

$$= \frac{\beta}{b^2}\left(\frac{3}{2}-2\right)\left(\ \right)^{3/2} = -\frac{\beta}{2b^2} = \left.+\frac{\hbar^2}{2mb^2}\right\}$$ From 1st 2 terms

$$\langle H \rangle = \frac{\hbar^2}{2mb^2} + \frac{c\,2\sqrt{2}}{\pi^{1/2}\, b}$$

c) $$\frac{d\langle H\rangle}{db} = -\frac{\hbar^2}{mb^3} - \frac{c\,2\sqrt{2}}{\pi^{1/2}\, b^2} = 0$$

$$\Rightarrow \quad b = -\frac{\hbar^2 \pi^{1/2}}{c\,2\sqrt{2}}$$

$$E_0 = \langle H \rangle \Big|_{min} = \frac{\hbar^2 c^2\, 8}{2m\, \hbar^4 \pi} - \frac{(c\,2\sqrt{2})^2}{\hbar^2 \pi}$$

$$= \frac{4c^2}{m\hbar^2\pi} - \frac{8c^2}{\hbar^2\pi}$$

$$=$$

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Problem 3: Artificial Atoms (10 points)

Modern techniques in nanotechnology research can create artificial atoms, man-made structures that confine electrons like real atoms but with properties that can be engineered. In this problem, consider a 2D atom (electrons tightly bound in the z-direction) with a parabolic potential in the x- and y-directions. The Hamiltonian is:

$$H_0 = \frac{p^2}{2m} + \frac{m\omega^2}{2}\left(x^2 + y^2\right). \tag{1}$$

Note: In solving this problem, you might want to use the standard operators:

$$a_x = \frac{1}{\sqrt{2}}\left(\frac{x}{\lambda} + i\frac{\lambda}{\hbar}p_x\right), \quad a_y = \frac{1}{\sqrt{2}}\left(\frac{y}{\lambda} + i\frac{\lambda}{\hbar}p_y\right) \tag{2}$$

and their Hermitian conjugates, where $\lambda = \sqrt{\frac{\hbar}{m\omega}}$.

a) What are the eigenenergies of this atom? What are the degeneracies of these energy levels? If the separation between adjacent levels is 20 meV (0.02 eV), approximately how large are the low-energy electron states in the atom (the radius)? (2 pts)

b) If the atom is put in a constant electric field, the Hamiltonian H_0 is perturbed by a potential:

$$H_1 = -eE_1 x \tag{3}$$

where E_1 is a constant (the electric field). Prove that to first order in the field, the energy levels of the atom do not change. (2 pts)

c) Next the atom is placed in a more complex field to study its properties. The new potential is:

$$H_2 = \frac{C_2}{\lambda^2}xy \tag{4}$$

To first order in C_2, what are the new eigenenergies of what were the first three energy levels of H_0? Show your work. (4 pts)

d) If a different perturbing potential:

check

$$H_3 = \frac{C_3}{\lambda^2}x^2 \tag{5}$$

is applied (rather than H_2), how would your answers to part (c) change? No computations should be necessary to answer this question. (2 pts)

P3

$$H_0 = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 (x^2+y^2)$$

a) $$E_n = (n_x+\tfrac{1}{2})\hbar\omega_x + (n_y+\tfrac{1}{2})\hbar\omega_y$$

$$= (n_x+n_y+1)\hbar\omega$$

$$= (n+\tfrac{1}{2})\hbar\omega$$

$$g_n = \binom{n+D-1}{n} = \binom{n+1}{n} = \frac{(n+1)!}{n!\,1!} = \frac{(n+1)n!}{n!} = n+1$$

$$\Delta E = 0.02\,eV = \hbar\omega \Rightarrow \omega = \frac{0.02}{\hbar}$$

Now, the energy of the state is equal to the pot. energy of the electron

so, for $n=0$

$$\hbar\omega = \frac{1}{2} m\omega^2 r^2$$

$$\frac{2\hbar}{m\omega} = r^2 \Rightarrow r = \left(\frac{2\hbar}{m\omega}\right)^{1/2}$$

is this right?

b) $$H_1 = -eE_1 X$$

$$E_n^{(1)} = \langle n | H_1 | n \rangle = -eE_1 \sqrt{\frac{\hbar}{2m\omega}} \langle n | a^+ + a | n \rangle$$

$$= 0$$

So, the energy level of the atom doesn't change

c) $$E_n^{(1)} = \langle n^{(0)} | H_2 | n^{(0)} \rangle$$

$$= \frac{C_2}{\lambda^2} \langle n_x^{(0)} n_y^{(0)} | \hat{x}\hat{y} | n_x^{(0)} n_y^{(0)} \rangle$$

$$= \underbrace{\frac{C_2}{\lambda^2}\left(\frac{\hbar}{2m\omega}\right)}_{\gamma} \langle n_x^{(0)} n_y^{(0)} | (a_x + a_x^+)(a_y + a_y^+) | n_x^{(0)} n_y^{(0)} \rangle$$

$$= \gamma \langle n_x^{(0)} n_y^{(0)} | a_x a_y + a_x a_y^+ + a_x^+ a_y + a_x^+ a_y^+ | n_x^{(0)} n_y^{(0)} \rangle$$

For, $n=0$

$$E_0^{(1)} = \gamma \langle 0\,0 | a_x a_y + a_x a_y^+ + a_x^+ a_y + a_x^+ a_y^+ | 0\,0 \rangle$$

$$= 0$$

$$E_0 = E_0^{(0)} + E_0^{(1)} = \hbar\omega$$

For, $n=1$ we have 2 states $|1\,0\rangle$ $|0\,1\rangle$

$$E_1^{(1)} = \gamma \{ 1 \langle 0\,1 | 0\,1 \rangle + 1 \langle 1\,0 | 1\,0 \rangle \}$$

$$= \frac{C_2 \hbar}{\lambda^2 m\omega} \qquad E_1 = E_1^{(0)} + E_1^{(1)} = \frac{3\hbar\omega}{2} + \frac{C_2 \hbar}{\lambda^2 m\omega}$$

For $n=2$ We have 3 degenerate states $|2\,0\rangle$ $|0\,2\rangle$ $|1\,1\rangle$

$$E_2^{(1)} = \gamma \{ \sqrt{1}\sqrt{2} \langle 1\,1|1\,1\rangle + \sqrt{1}\sqrt{2} \langle 1\,1|1\,1\rangle + \sqrt{2}\sqrt{1} \langle 0\,2|0\,2\rangle$$

$$+ \sqrt{2}\sqrt{1} \langle 2\,0|2\,0\rangle \}$$

$$= \frac{c_2}{\lambda^2} \frac{2\sqrt{2}\hbar}{m\omega}$$

d) $$H_3 = \frac{c_3}{\lambda^2} x^2$$

$$E_0^{(1)} = \frac{c_3}{\lambda^2} \left(\frac{\hbar}{2m\omega}\right)$$

$$E_1^{(1)} = 0$$

$$E_2^{(1)} = 0$$

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Problem 4: 3-d central-force problem (10 points)

A particle of mass m and spin $s = 0$ has a short-range potential energy $V(r)$. The particle is in a stationary state with Hamiltonian eigenfunction

$$\psi_E(\boldsymbol{r}) = N \frac{1}{r} \left(e^{-\alpha r} - e^{-\beta r} \right), \tag{6}$$

where N is a normalization constant (which you need not determine), and α and β are real numbers such that $\beta > \alpha$.

1. Is the orbital angular momentum of the particle sharp in this state? (That is, does L^2 have zero uncertainty?) If not, explain why not. If so, justify your answer and give the value of L^2 for this state. (4 pts)
2. What is the stationary-state energy of this state? (4 pts)
3. What is the potential energy $V(r)$? (2 pts)

P4.

1) $\psi_E(r) = N\frac{1}{r}(e^{-\alpha r} - e^{-\beta r})$ is the eigenfunc of the Hamiltonian

Since, $[L^2, H] = 0$

$$(\Delta L^2) = \sqrt{\langle L^4\rangle - \langle L^2\rangle^2} = 0$$

$L^2 = \hbar^2 l(l+1) = 0$ Since $l = 0$

2) $H\psi_E(r) = E\psi_E(r)$

$$H = -\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{L^2}{2mr^2} + V(r)$$

In this case $L^2 = 0$

$$H = -\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + V(r)$$

Now,

$$r^2\frac{\partial\psi_E(r)}{\partial r} = r^2\frac{\partial}{\partial r}\left\{\frac{N}{r}(e^{-\alpha r} - e^{-\beta r})\right\}$$

$$= r^2 N\left\{-\frac{1}{r^2}(e^{-\alpha r} - e^{-\beta r}) + \frac{1}{r}(-\alpha e^{-\alpha r} + \beta e^{-\beta r})\right\}$$

$$= N\left\{-(e^{-\alpha r} - e^{-\beta r}) + r(-\alpha e^{-\alpha r} + \beta e^{-\beta r})\right\}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\{\ \} = \frac{N}{r^2}\left[\cancel{\alpha e^{-\alpha r}} - \cancel{\beta e^{-\beta r}} - \cancel{\alpha e^{-\alpha r}} + \cancel{\beta e^{-\beta r}} + r(\alpha^2 e^{-\alpha r} - \beta^2 e^{-\beta r})\right]$$

$$= N\frac{1}{r}(\alpha^2 e^{-\alpha r} - \beta^2 e^{-\beta r})$$

NOW,

$$\frac{N}{r}\left(\alpha^2 e^{-\alpha r} - \beta^2 e^{-\beta r}\right) + V(r)\Psi(r) = E\Psi(r)$$

$$\Rightarrow \underbrace{\frac{N}{r}\left(e^{-\alpha r} - e^{-\beta r}\right)}_{\Psi(r)} \left\{ -\frac{\hbar^2}{2m}\frac{\alpha^2 e^{-\alpha r} - \beta^2 e^{-\beta r}}{e^{-\alpha r} - e^{-\beta r}} \right\} = (E - V)\Psi(r)$$

$$\Rightarrow E - V(r) = -\frac{\hbar^2}{2m}\frac{\alpha^2 e^{-\alpha r} - \beta^2 e^{-\beta r}}{e^{-\alpha r} - e^{-\beta r}} = -\frac{\hbar^2}{2m}\frac{\alpha^2 - \beta^2 e^{-(\beta-\alpha)r}}{1 - e^{-(\beta-\alpha)r}}$$

Since, the potential is short ranged

$$V(r) \to 0 \text{ as } r \to \infty$$

$$\Rightarrow E = -\frac{\hbar^2\alpha^2}{2m}$$

3) $$V(r) = -\frac{\hbar^2\alpha^2}{2m} - \frac{\alpha^2 - \beta^2 e^{-(\beta-\alpha)r}}{1 - e^{-(\beta-\alpha)r}}$$

$$= \frac{-\hbar^2\alpha^2 + \hbar^2\alpha^2 e^{-(\beta-\alpha)r}}{2m\left(1 - e^{-(\beta-\alpha)r}\right)}$$

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Problem 5: Quantum statistics (10 points)

1. Write down the energy eigenvalues and wave functions for a particle of mass m in an infinite square well, with $V = 0$ for $-L/2 < x < L/2$ and $V = \infty$ for $|x| > L/2$. (2 pts)

2. What is the ground state energy and wave-function if 2 identical non-interacting bosons are in the well? (4 pts)

3. What is the ground state energy and wave-function if 2 identical non-interacting spin-up fermions are in the well? (4 pts)

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Problem 6: Spin $\frac{1}{2}$ System (10 points)

Consider a spin $\frac{1}{2}$ particle in the state space E_s. This space can be spanned by the 2 eigenvectors of S_x, S_y, or S_z, the components of the spin operator $S = S_x\hat{i} + S_y\hat{j} + S_z\hat{k}$. The matrix representation of S_x, S_y and S_z in the eigenbasis $|+\rangle_z$, $|-\rangle_z$ of S_z are given below:

$$S_x = \hbar/2\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad S_y = \hbar/2\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad S_z = \hbar/2\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

where $S_z|+\rangle_z = \hbar/2|+\rangle_z$ and $S_z|-\rangle_z = -\hbar/2|-\rangle_z$.

Assume that the state of the system at time $t = 0$ is: $|\Psi(0)\rangle = |-\rangle_z$.

a) If the observable S_x is measured at time $t = 0$, what results can be found and with what probabilities? (1 pt)

Now assume that a magnetic field is applied in the x **direction**: $\vec{B} = B_0\hat{i}$. The original wave function $|\Psi(0)\rangle = |-\rangle_z$ is allowed to evolve in time. The Hamiltonian governing the evolution is:

$$H_{spin} = \vec{S}\cdot\vec{B}$$

b) Set up the time evolution operator for this system, $U(t, 0)$. (1 pt)

c) Find $|\Psi(t)\rangle$, the wave function at a later time t. (1 pt)

d) At time $t > 0$ after $|\Psi(0)\rangle$ has evolved, S_x is measured. What is the probability of obtaining $+\hbar/2$? Is your answer time dependent or time independent? Explain correctly for credit. (1 pt)

e) Now let $|\Psi(0)\rangle$ evolve again and measure S_z at time t. Determine the probability of measuring S_z at time t and obtaining $-\hbar/2$. Is your answer time dependent or time independent? Explain correctly for credit. (1 pt)

f) Without explicitly finding the probabilities, discuss whether you expect the following probabilities to be equal or not. Give a brief explanation of your reasoning for any credit. The symbol $P_{|\Psi(t)\rangle}(a, c)$ represents the probability of starting with an ensemble in the state $|\Psi(t)\rangle$, measuring A first and getting eigenvalue "a" and then measuring C and getting eigenvalue "c". Assume that the eigenvalues of H_{spin} are E_+ and E_-. (1 pt)

i) Is $P_{|\Psi(0)\rangle}(+\hbar/2 \text{ for } S_y, -\hbar/2 \text{ for } S_x) = P_{|\Psi(0)\rangle>}(-\hbar/2 \text{ for } S_x, +\hbar/2 \text{ for } S_y)$? All measurements are taken at $t = 0$, i.e. the second measurement is taken immediately after the first measurement in each case. (1 pt)

ii) Is $P_{|\Psi(0)\rangle}(E_+, -\hbar/2 \text{ for } S_x) = P_{|\Psi(0)\rangle}(-\hbar/2 \text{ for } S_x,\ E_+)$? The first measurement in each case is taken at $t = 0$; the second measurement is taken immediately after the first measurement in each case. (1 pt)

iii) Is the probability $P_{|\Psi(0)\rangle}(+\hbar/2 \text{ for } S_x \text{ at } t, -\hbar/2 \text{ for } S_y \text{ at } t')$ time dependent or time independent in regards to the time t of the first measurement? Same question for the time t' of the second measurement. Discuss your reasoning in each case. (2 pts)

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