E & M Qualifier

August XX, 2010

To insure that the your work is graded correctly you MUST:

1. use only the blank answer paper provided,
2. write only on one side of the page,
3. put your alias on every page,
4. put the problem # on every page,
5. start each problem by stating your units e.g., SI or Gaussian,
6. number every page starting with 1 for each problem,
7. put the total # of pages you use for that problem on every page,
8. staple your exam when done.

Use only the reference material supplied (Schaum's Guides).

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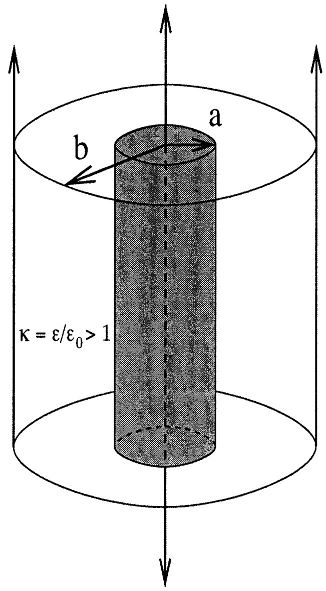


$\rho_{out} = \frac{\lambda L}{\pi a^2 L} = \frac{\lambda}{\pi a^2}$

$\rho_{in} = \frac{\lambda L}{\pi r^2 L} = \frac{\lambda}{\pi r^2}$

1. A very long conducting wire of radius a, carrying free positive charge per unit length λ, is surrounded by a dielectric coating of outside radius b and relative dielectric constant $\kappa = \epsilon/\epsilon_0$.

   (a) {2 pts} Find the displacement vector **D** everywhere.

   (b) {2 pts} Find the electric field **E** everywhere.

   (c) {2 pts} Find the polarization density **P** everywhere.

   (d) {2 pts} Find the bound volume charge density ρ_b and the bound surface charge density σ_b everywhere.

   (e) {2 pts} Show that the total charge densities, bound and free, produce the same **E** found in (a).

Problem 1 (Gaussian)

(a) $\underline{a < r < b}$

$$\int \bar{D} \cdot d\bar{a} = 4\pi Q_{f\,enc}$$

$$\Rightarrow \quad D(2\pi r L) = 4\pi \lambda L$$

$$\Rightarrow \quad \bar{D} = \frac{2\lambda}{r} \hat{r}$$

$\underline{\underline{r > b}}$

$$\bar{D} = \frac{2\lambda}{r} \hat{r}$$

b) $\underline{a < r < b}$

$$\bar{E} = \frac{2\lambda}{\epsilon r} \hat{r}$$

$\underline{\underline{r > b}}$

$$\bar{E} = \frac{2\lambda}{r} \hat{r}$$

c) $\underline{\underline{a < r < b}}$

$$\bar{P} = \frac{1}{4\pi} \frac{2\lambda}{r} \left(1 - \frac{1}{\epsilon} \right) \hat{r}$$

$\underline{\underline{r > b}}$ $\quad \bar{P} = 0$

d) $\underline{a < r < b}$

$$\rho_b = - \nabla \cdot \bar{P} = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{1}{r} \right) \frac{1}{4\pi} 2\lambda \left(1 - \frac{1}{\epsilon} \right)$$

$$= 0$$

$\underline{r > b}$ $\quad \rho_b = 0$

$\underline{r = a}$ $\quad \sigma_b = \bar{P}.\hat{r} = -\frac{1}{4\pi}\frac{2\lambda}{a}\left(1-\frac{1}{\epsilon}\right)$

$\underline{r = b}$ $\quad \sigma_b = \bar{P}.\hat{r} = \frac{1}{4\pi}\frac{2\lambda}{b}\left(1-\frac{1}{\epsilon}\right)$

d) $\underline{a < r < b}$

$$E\, 2\pi r L = 4\pi\lambda L - \frac{2\lambda}{a}\left(1-\frac{1}{\epsilon}\right)2\pi a L$$

$$= 4\pi\lambda L\left(1 - 1 + \frac{1}{\epsilon}\right)$$

$$\Rightarrow \bar{E} = \frac{2\lambda}{\epsilon r}\hat{r}$$

$\underline{r > b}$

$$E\, 2\pi r = 4\pi\lambda L$$

$$\bar{E} = \frac{2\lambda}{r}\hat{r}$$

2. (a) {2 pts} In a homogeneous, linear and isotropic conducting material whose electromagnetic properties (at low frequencies) are described by constant values of the permittivity, pearmeability, and conductivity respectively ϵ, μ, and σ, show that Maxwell's equations require that the electric field satisfy

$$\nabla^2 \mathbf{E} - \epsilon\mu \frac{\partial^2 \mathbf{E}}{\partial^2 t} - \sigma\mu \frac{\partial \mathbf{E}}{\partial t} = 0, \qquad (SI)$$

$$\nabla^2 \mathbf{E} - \frac{\epsilon\mu}{c^2} \frac{\partial^2 \mathbf{E}}{\partial^2 t} - \frac{4\pi\sigma\mu}{c^2} \frac{\partial \mathbf{E}}{\partial t} = 0. \qquad (Gaussian)$$

(b) {2 pts} Given a plane polarized plane wave of angular frequency ω whose electric field is of the form

$$\mathbf{E}(z,t) = \text{Real}\left\{\hat{\imath} E_0 e^{i(kz-\omega t)}\right\},$$

evaluate k^2 as a function of ϵ, μ, σ, and ω.

(c) {2 pts} Find the real and imaginary parts of k assuming $\sigma >> \omega\epsilon$.

(d) {2 pts} Using your results from (c) find the skin depth δ of the conductor. The skin depth is defined by the depth at which the wave's amplitude decreases by e^{-1}, i.e.,

$$\frac{|\mathbf{E}(z+\delta,t)|}{|\mathbf{E}(z,t)|} = \frac{1}{e}$$

(e) {2 pts} Using Maxwell's equations, find the magnetic field $\mathbf{H}(x,t)$ associated with $\mathbf{E}(z,t)$ given in (b) and discuss their phase difference when $\sigma >> \omega\epsilon$.

Prob 2 (Gaussian)

a) $\nabla \cdot \bar{E} = \frac{4\pi \underbrace{\rho_f}_{\overset{?}{=}0}}{\epsilon}$ $\qquad \nabla \wedge \bar{E} = -\frac{1}{c}\frac{\partial \bar{B}}{\partial t}$

$\nabla \cdot \bar{B} = 0$ $\qquad \nabla \wedge \frac{B}{\mu} - \frac{1}{c}\epsilon\frac{\partial E}{\partial t} = \frac{4\pi}{c}\underbrace{J_f}_{=\sigma E}$

σ ← conductivity

$$\Rightarrow \frac{\partial}{\partial t}\left(\nabla \wedge \frac{B}{\mu}\right) - \frac{\epsilon}{c}\frac{\partial^2 E}{\partial t^2} = \frac{4\pi\sigma}{c}\frac{\partial E}{\partial t}$$

$$\Rightarrow \underbrace{\nabla \wedge \left(\frac{\partial B}{\partial t}\right)} - \frac{\epsilon\mu}{c}\frac{\partial^2 E}{\partial t^2} = \frac{4\pi\sigma\mu}{c}\frac{\partial E}{\partial t}$$

$$\underbrace{-c\nabla \wedge (\nabla \wedge E)}$$

$$= \nabla(\nabla \cdot E) - \nabla^2 E$$

$$\frac{\partial \rho_f}{\partial t} = -\nabla \cdot \bar{J} = -\sigma(\nabla \cdot \bar{E}) = -\frac{4\pi\sigma}{\epsilon}\rho_f$$

$$\Rightarrow \rho_f(t) = A e^{-\left(\frac{4\pi\sigma}{\epsilon}\right)t} \Rightarrow \rho_f(t) = \rho_f(0)\, e^{-\frac{4\pi\sigma}{\epsilon}}$$

Assuming no charge density $\rho_f = 0 \Rightarrow \nabla \cdot \bar{E} = 0$

$$\Rightarrow \nabla^2 E - \frac{\epsilon\mu}{c^2}\frac{\partial^2 \bar{E}}{\partial t^2} - \frac{4\pi\sigma\mu}{c^2}\frac{\partial \bar{E}}{\partial t} = 0$$

b) $E(z,t) = \mathrm{Re}\left\{\hat{x}\, E_0\, e^{i(kz-\omega t)}\right\}$

using $$\nabla^2 \bar{E} - \frac{\epsilon\mu}{c^2}\frac{\partial^2 \bar{E}}{\partial t^2} - \frac{4\pi\sigma\mu}{c^2}\frac{\partial \bar{E}}{\partial t} = 0$$

$$\Rightarrow (ik)^2 E_0 e^{i(kz-\omega t)} - \frac{\epsilon\mu}{c^2}(-i\omega)^2 E_0 e^{i(kz-\omega t)} - \frac{4\pi\sigma\mu}{c^2}(-i\omega) E_0 e^{i(kz-\omega t)} = 0$$

$$\Rightarrow -k^2 + \frac{\epsilon\mu\omega^2}{c^2} + i\,\frac{4\pi\sigma\mu}{c^2}\,\omega = 0$$

$$\Rightarrow k^2 = \frac{\epsilon\mu\omega^2}{c^2} + i\,\frac{4\pi\sigma\mu}{c^2}\,\omega \Rightarrow k = \frac{\omega}{c}\sqrt{\epsilon\mu}\left[1 + i\,\frac{4\pi\sigma}{\omega\epsilon}\right]^{1/2}$$

c) $$\tilde{k} = \left\{\frac{\epsilon\mu\omega^2}{c^2} + i\,\frac{4\pi\sigma\mu}{c^2}\,\omega\right\}^{1/2}$$

$\tilde{k}\tilde{k}^* = \{$

$\tilde{k} = k + i\kappa$

$\tilde{k}\tilde{k}^* = (k+i\kappa)(k-i\kappa)$

$= k^2 + i\kappa k - ik\kappa + \kappa^2$

$\tilde{k}^2 = k^2 + \kappa^2$

c) $K = \frac{\omega}{c}\sqrt{\epsilon\mu}\left[1 + i\,\frac{4\pi\sigma}{\omega\epsilon}\right]^{1/2}$

If $\sigma \gg \omega\epsilon$

then $K_{real} \approx 0$

$$K_{imag} \approx \underbrace{\frac{\omega}{c}\sqrt{\epsilon\mu}\left(\frac{4\pi\sigma}{\omega\epsilon}\right)^{1/2}}_{\alpha}$$

d) $$\frac{|E(z+\delta,t)|}{|E(z,t)|} = \frac{1}{e}$$

$$E(z+\delta,t) = \hat{x}E_0\, e^{i(i\alpha(z+\delta)-i\omega t}$$

$$= \hat{x}\, e^{-\alpha-i\omega t}\, e^{-\alpha\delta}$$

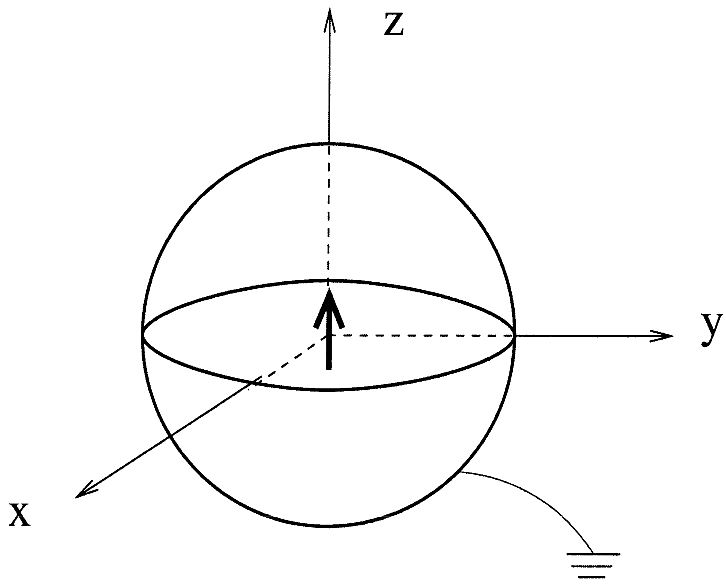
$$E(z,t) = \hat{x}\, e^{-\alpha-i\omega t}$$

$$\frac{|E(z+\delta,t)|}{|E(z,t)|} = \frac{1}{e}$$

$$\Rightarrow \quad e^{-\alpha\delta} = e^{-1}$$

$$\Rightarrow \quad \delta = \frac{1}{\alpha}$$

3. A point electric dipole with dipole moment $\mathbf{p} = p_0\hat{k}$ is located at the center of a hollow, grounded, conducting sphere.



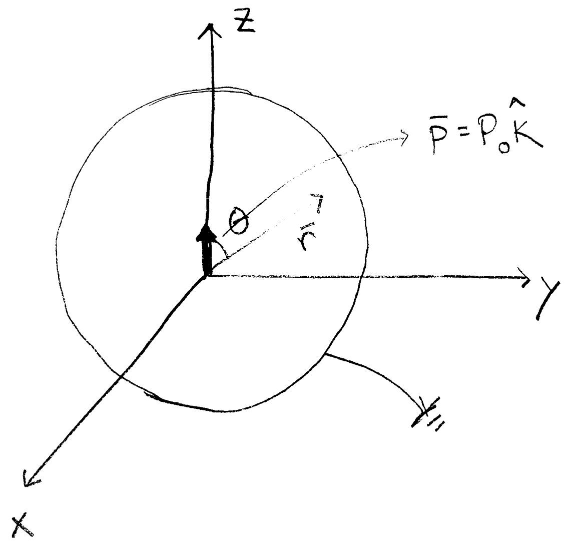
(a) {2 pts} What are the boundary conditions satisfied by the electric field and electric potential in this problem?

(b) {5 pts} Compute the electrostatic potential inside the sphere.

(c) {3 pts} Compute the charge density σ on the inside surface on the grounded sphere.

Prob 3 (Gaussian)

a) $E^{\perp}_{outside} - E^{\perp}_{inside} = 4\pi\sigma$



$\Phi_{in}(r=R) = \Phi_{out}(r=R) = 0$

b) $$\Phi^{dipole}_{in}(r,\theta) = \frac{\bar{p}\cdot(\bar{r}-\bar{r}')}{|\bar{r}-\bar{r}'|^3} = \frac{\bar{p}\cdot\bar{r}}{r^3} = \frac{p_0 r\cos\theta}{r^3}$$

$$= \frac{p_0\cos\theta}{r^2}$$

$$\Phi^{INSIDE}_{sphere}(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

$$\rightsquigarrow \quad \Phi_{tot}(r=R) = \Phi_{out}(r=R) = 0$$

$$\Rightarrow \sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = -\frac{p_0}{R^2} P_1(\cos\theta)$$

$$\Rightarrow A_1 = -\frac{p_0}{R^3} \quad \& \quad \sum_{\substack{l=0 \\ (l\neq 1)}}^{\infty} A_l = 0$$

$$\Rightarrow \Phi^{inside}_{tot}(r,\theta) = -\frac{p_0}{R^3} r\cos\theta + \frac{p_0\cos\theta}{r^2} = \frac{p_0\cos\theta}{r^2}\left(1 - \frac{r^3}{R^3}\right)$$

c) $E^{\perp}_{outside} - E^{\perp}_{inside} = 4\pi\sigma$

$$\Rightarrow -\frac{\partial \Phi^{outside}}{\partial r}\Big|_{r=R} + \frac{\partial \Phi^{inside}}{\partial r}\Big|_{r=R} = 4\pi\sigma$$

$$\Rightarrow \sigma = +\frac{1}{4\pi}\frac{\partial \Phi}{\partial r}\Big|_{r=R}$$

$$= +\frac{1}{4\pi}\frac{\partial}{\partial r}\left\{\frac{P_0\cos\theta}{r^2} - \frac{P_0\cos\theta}{R^3}\cdot r\right\}$$

$$= +\frac{1}{4\pi}\left\{\frac{(-2)P_0\cos\theta}{r^3} - \frac{P_0\cos\theta}{R^3}\right\}\Big|_{r=R}$$

$$= \frac{-3P_0\cos\theta}{4\pi R^3}$$

4. (25 points) A 50 MeV electron ($m\gamma c^2 = 50$ MeV, $mc^2 = 0.5$ MeV) moving along the z-axis is decelerated and brought to a stop after traveling 10 cm in a uniform electric field $\mathbf{E} = E_0\hat{\mathbf{k}}$. (Recall $\gamma \equiv 1/\sqrt{1-\frac{v^2}{c^2}}$.)

   (a) {3 pts} Compute $\gamma(t)$ assuming the electron starts its deceleration at t=0.

   (b) {3 pts} How long does it take the electron to stop?

   (c) {3 pts} Compute the total energy radiated by the electron during the 10 cm stopping process.

   (d) {1 pts} What fraction of the electrons initial energy was lost to radiation?

Hint: The general Larmor formula for power radiated by an accelerating point charge is

$$P(t) = \frac{2}{3}\frac{q^2}{4\pi\epsilon_0 c}\gamma^6[(\dot{\vec{\beta}})^2 - (\vec{\beta}\times\dot{\vec{\beta}})^2], \qquad (SI)$$

$$P(t) = \frac{2}{3}\frac{q^2}{c}\gamma^6[(\dot{\vec{\beta}})^2 - (\vec{\beta}\times\dot{\vec{\beta}})^2]. \qquad (Gaussian)$$

It might be useful to use $\dot{(\gamma\beta)} = \gamma^3\dot{\beta}$.

$$\begin{aligned} 1e &= 4.8\times 10^{-10}\text{statcoul} = 1.6\times 10^{-19}\text{coul}, \\ 1eV &= 1.6\times 10^{-12}\text{ergs} = 1.6\times 10^{-19}\text{J}. \end{aligned}$$

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5. A plane-polarized electromagnetic wave traveling in vacuum is observed in the lab frame as

$$\mathbf{E} = \mathrm{Real}\Big\{ E_0\, \hat{\imath}\, e^{i(kz-\omega t)} \Big\},$$

$$\mathbf{B} = \mathrm{Real}\Big\{ B_0\, \hat{\jmath}\, e^{i(kz-\omega t)} \Big\},$$

where $k = \omega/c$ and

$$\begin{aligned} B_0 &= E_0/c, &\qquad (SI) \\ B_0 &= E_0. &\qquad (Gaussian) \end{aligned}$$

A relativistic particle moving in the z-direction ($\mathbf{v} = v_0\hat{k}$, where $\gamma >> 1$) encounters this wave. For this problem you are to find the form of the wave in the rest frame of the particle at the instant the wave is first encountered (before the particle's velocity is changed because of an interaction with the wave).

(a) {2 pts} Start by combining **E** and **B** into a single 4-tensor $F^{\alpha\beta}(x)$. This includes writing $(kz - \omega t) = \pm k_\alpha x^\alpha$. The sign $\pm$ depends on your choice of Lorentz metrics $(-1, +, +, +)$ or $(1, -1, -1, -1)$. State which you are using.

confused with $k_\alpha x^\alpha$

(b) {2 pts} Give the Lorentz transformation $L^\alpha{}_\beta$ that transforms the lab frame into the particle's rest frame $x'^\alpha = L^\alpha{}_\beta\, x^\beta$.

(c) {2 pts} Apply your Lorentz transformation to $F^{\alpha\beta}(x)$ to find $F'^{\alpha\beta}(x')$, the electromagnetic 4-tensor in the particles rest frame.

(d) {2 pts} From your results in (c) give the 3-dimensional propagation direction of the wave in the particle's frame and the $\mathbf{E}'(x')$ and $\mathbf{B}'(x')$ fields.

(e) {2 pts} Compare the amplitudes and frequency of the wave as seen by the particle in its rest frame with those seen by a lab observer?

5. (Gaussian)

$$\bar{E} = \text{Real}\left\{E_0\,\hat{i}\,e^{i(kz-\omega t)}\right\} = E_0\cos(kz-\omega t)\,\hat{x}$$

$$\bar{B} = \text{Real}\left\{B_0\,\hat{j}\,e^{i(kz-\omega t)}\right\} = E_0\cos(kz-\omega t)\,\hat{y}$$

$$g_{\alpha\beta} = (1,-1,-1,-1) \qquad x^\alpha = \begin{pmatrix} ct \\ 0 \\ 0 \\ z \end{pmatrix}$$

$$K_\alpha x^\alpha = k_0 x^0 + \underbrace{k_1 x^1}_{=0} + \underbrace{k_2 x^2}_{=0} + k_3 x^3 \qquad K_\alpha = \begin{pmatrix} \frac{\omega}{c} \\ 0 \\ 0 \\ -K \end{pmatrix}$$

$$= \omega t - Kz$$

$$\Rightarrow \quad -\underbrace{K_\alpha x^\alpha}_{} = (kz-\omega t)$$

↑scalar invariant

$$F^{\alpha\beta}(x) = \begin{pmatrix} 0 & -E_0\cos(kz-\omega t) & 0 & 0 \\ E_0\cos(kz-\omega t) & 0 & 0 & E_0\cos(kz-\omega t) \\ 0 & 0 & 0 & 0 \\ 0 & -E_0\cos(kz-\omega t) & 0 & 0 \end{pmatrix}$$

$$F^{\alpha\beta}(x^\mu) = \begin{pmatrix} 0 & -E_0\cos(-K_\alpha x^\alpha) & 0 & 0 \\ E_0\cos(-K_\alpha x^\alpha) & 0 & 0 & E_0\cos(-K_\alpha x^\alpha) \\ 0 & 0 & 0 & 0 \\ 0 & -E_0\cos(-K^\alpha x_\alpha) & 0 & 0 \end{pmatrix}$$

b) $x'^{\alpha} = L^{\alpha}_{\beta}\, x^{\beta} \Rightarrow x^{\beta} = (L^{-1})^{\alpha}_{\beta}\, x'^{\alpha}$

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} ct \\ 0 \\ 0 \\ z \end{pmatrix} = \begin{pmatrix} \gamma ct - \gamma\beta z \\ 0 \\ 0 \\ -\gamma\beta ct + \gamma z \end{pmatrix}$$

$\therefore$ $ct' = \gamma(ct - \beta z)$ | $ct = \gamma(ct' + \beta z')$

$z' = \gamma(-\beta ct + z)$ | $z = \gamma(\beta ct' + z')$

c) $F'^{\alpha\beta}(x) = L^{\alpha}_{\gamma} L^{\beta}_{\sigma} F^{\gamma\sigma}(x)$

$$\bar{E} = \bar{E}'_{\parallel} + \gamma(\bar{E}'_{\perp} - \bar{\beta} \wedge \bar{B}')$$

$$\bar{B} = \bar{B}'_{\parallel} + \gamma(\bar{B}'_{\perp} + \bar{\beta} \wedge \bar{E}')$$

Hence,

$$\bar{E}' = \bar{E}_{\parallel} + \gamma(\bar{E}_{\perp} + \bar{\beta} \wedge \bar{B})$$

$$\bar{B}' = \bar{B}_{\parallel} + \gamma(B_{\perp} - \bar{\beta} \wedge E)$$

$$\bar{\beta} = \begin{pmatrix} 0 \\ 0 \\ \beta \end{pmatrix}$$

So,

$$\bar{E}' = \gamma\left[\begin{pmatrix} E_0 \cos(Kz-\omega t) \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \beta \end{pmatrix} \wedge \begin{pmatrix} 0 \\ E_0 \cos(Kz-\omega t) \\ 0 \end{pmatrix}\right]$$

$$= \gamma\left[\begin{pmatrix} E_0 \cos(Kz-\omega t) \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \beta E_0 \cos(Kz-\omega t) \\ 0 \\ 0 \end{pmatrix}\right]$$

$$E'(x) = \gamma(1+\beta) E_0 \cos(Kz-\omega t) \quad \hat{x}$$

$$E'(x') = \gamma(1+\beta) E_0 \cos\left\{K\gamma(\beta c t' + z') - \frac{\omega}{c}\gamma(ct' + \beta z')\right\}$$

$$\bar{B} = \gamma\left[\begin{pmatrix} 0 \\ E_0 \cos(Kz-\omega t) \\ 0 \end{pmatrix} - \underbrace{\begin{pmatrix} 0 \\ 0 \\ \beta \end{pmatrix} \wedge \begin{pmatrix} E_0 \cos(Kz-\omega t) \\ 0 \\ 0 \end{pmatrix}}_{\begin{pmatrix} 0 \\ \beta E_0 \cos(Kz-\omega t) \\ 0 \end{pmatrix}}\right]$$

$$= \gamma(1-\beta) \cos(Kz-\omega t) \hat{y}$$

$$\bar{B}'(x') = \gamma(1-\beta) \cos\left(K\gamma(\beta c t' + z') - \frac{\omega}{c}\gamma(ct' + \beta z')\right)$$

c) Now,

$$E'(x') = \gamma(1+\beta)E_0 \cos\left\{ z'\left(k\gamma - \omega\gamma\frac{\beta}{c}\right) - (\omega\gamma - k\gamma\beta c)t' \right\} \hat{x}$$

$$k'_\alpha x'_\alpha = (k'_0 \; -k'_1 \; -k'_2 \; -k'_3) \begin{pmatrix} ct' \\ 0 \\ 0 \\ z' \end{pmatrix}$$

$$= \omega' t' - k'_3 z'$$

So, $\vec{k} = k\hat{z}$ $\qquad \vec{E}' = E'\hat{x}$

$\qquad\qquad\qquad\qquad \vec{B}' = B'\hat{y}$

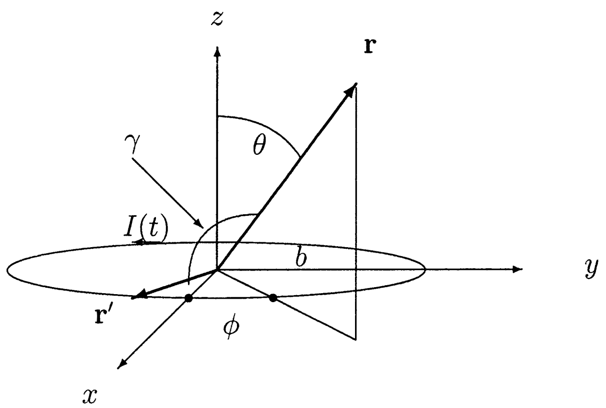
Since, $k'z' = kz$

d)

$$\Rightarrow \omega' = \omega\gamma - k\gamma\beta c$$

$$= \gamma(\omega - kv)$$

$$\Rightarrow k' = \gamma\left(k - \frac{\omega\beta}{c}\right)$$



6. A circular current loop of radius b lies in the x-y plane and is centered on the origin. If the current varies harmonically with time as $I(t) = I_0 \cos(\omega t)$, use the following to carry out steps (a) through (e):

$$\begin{aligned} \mathbf{r}' &= b\{\sin\theta'(\cos\phi'\hat{\imath} + \sin\phi'\hat{\jmath}) + \cos\theta'\hat{k}\} \\ &= b(\cos\phi'\hat{\imath} + \sin\phi'\hat{\jmath}), \\ \mathbf{r}\cdot\mathbf{r}' &= b\,r\,\cos\gamma \\ &= b\,r\{\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')\} \\ &= b\,r\,\sin\theta\cos(\phi-\phi'). \end{aligned}$$

In the above $\mathbf{r}'$ is a point on the current loop with spherical-polar coordinates $r' = b$, $\theta' = \pi/2$, $0 \le \phi' \le 2\pi$, and γ is the angle between $\mathbf{r}$ and $\mathbf{r}'$.

(a) {2 pts} Compute the time dependent magnetic dipole moment $\mathbf{m}(t)$ of the current loop. Recall that $\mathbf{m}_{Gaussian} = \mathbf{m}_{SI}/c$.

(b) {2 pts} Give an integral expression for the retarded vector potential $\mathbf{A}(t, \mathbf{r})$.

(c) {2 pts} Approximate the integral found in (b) for $\mathbf{A}$ assuming $b << r$ and $b << c/\omega$. If you have made no mistakes your answer should agree with the potential for a point magnetic dipole, i.e., with:

$$\begin{aligned} \mathbf{A} &= \frac{\mu_0}{4\pi}\nabla\times\left\{\frac{\mathbf{m}(t-r/c)}{r}\right\}, && SI \\ \mathbf{A} &= \nabla\times\left\{\frac{\mathbf{m}(t-r/c)}{r}\right\}. && Gaussian \end{aligned}$$

(d) {2 pts} From your results for (c) or from the point magnetic dipole result, compute the radiation (far field) part of $\mathbf{E}$ by assuming $b << c/\omega << r$.

(e) {2 pts} Using only the radiation part, i.e., the part $\propto 1/r$, of $\mathbf{E}$ and $\mathbf{B}$, and the Poynting vector, compute the time averaged electromagnetic energy flux radiated away by the dipole as a function of the spherical polar coordinates (r, θ, ϕ). Recall that

$$\begin{aligned} \mathbf{B} &= \frac{1}{c}\hat{r}\times\mathbf{E}, && (SI) \\ \mathbf{B} &= \hat{r}\times\mathbf{E}, && (Gaussian) \end{aligned}$$

for radiation coming from a source at the origin.

Prob 6 (Gaussian)

$$\hat{\phi}' = -\sin\theta' \sin\phi' \hat{x} + \sin\theta' \cos\phi' \hat{y} = -\sin\phi' \hat{x} + \cos\phi' \hat{y}$$

a) $$\bar{m} = \frac{1}{2c} \int \bar{r}' \times \bar{J}(t', r') \, d^3 r'$$

$$= \frac{1}{2c} \int b (\cos\phi' \hat{x} + \sin\phi' \hat{y}) \times J(t', r') (-\sin\phi' \hat{x} + \cos\phi' \hat{y}) \, d^3 r'$$

$$= \frac{b}{2c} \int J(t', r') (\cos^2\phi' \hat{z} + \sin^2\phi' \hat{z}) \, d^3 r' = \frac{\hat{z} b}{2c} \underbrace{\int \bar{J}(t', r') d^3 r'}_{I(t)}$$

$$\bar{m}(t) = \hat{z} \frac{b}{2c} I_0 \cos\omega t \, \underbrace{\delta(x-b)\delta(y-b)\delta(z)}_{= \frac{\delta(r-b)\delta(\theta-\pi/2)}{2\pi r}}$$

$$= \hat{z} \frac{b}{2c} I_0 \cos\omega t \, (2\pi b) = \frac{\pi b^2}{c} I(t) \hat{z}$$

b) $$\Box A^\beta = \frac{4\pi}{c} J^\beta$$

$$\Rightarrow A^\beta(x) = \int_{(\infty)^4} G^{ret}(x, x') \frac{4\pi}{c} J^\beta \, d^4 x'$$

$$\Rightarrow \bar{A}(t, r) = \int \frac{\delta(x^0 - x^{0\prime} - |\bar{r} - \bar{r}'|)}{4\pi |\bar{r} - \bar{r}'|} \frac{4\pi}{c} \bar{J}(r') \, c\,dt' \, d^3 r'$$

$$= \frac{1}{c} \int \frac{\delta(t - t' - \frac{|\bar{r} - \bar{r}'|}{c})}{|\bar{r} - \bar{r}'|} \bar{J}(t', r') \, dt' \, d^3 r'$$

$$\Rightarrow \bar{A}(t,r) = \frac{1}{c}\int_{(\infty)^3} \frac{\bar{J}\left(t - \frac{|\bar{r}-\bar{r}'|}{c}\right)}{|\bar{r}-\bar{r}'|} d^3r'$$

c).
$$\left(|\bar{r}-\bar{r}'|\right)^{-1} = \left(\sqrt{r^2 - 2\bar{r}\cdot\bar{r}' + r'^2}\right)^{-1} = \left(r^2 - 2br\sin\theta\cos(\phi-\phi') + b^2\right)^{-1/2}$$

$$= \frac{1}{r}\left(1 - 2\frac{b}{r}\sin\theta\cos(\phi-\phi') + \frac{b^2}{r^2}\right)^{-1/2}$$

$$\approx \frac{1}{r}$$

$$\bar{A}(t,\bar{r}) = \frac{1}{c}\int \frac{J(t-\frac{r}{c})}{r}\hat{\phi}\, d^3r'$$

$$= \frac{1}{c}\int \frac{J(t-\frac{r}{c})}{r}\left(-\sin\phi'\hat{x} + \cos\phi'\hat{y}\right) d^3r'$$

d) $$\bar{A}(t,\bar{r}) = \bar{\nabla} \times \left\{ \frac{\bar{m}(t-r/c)}{r} \right\}$$

$$= \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ \frac{m^z(t-r/c)}{r} \end{pmatrix}$$

$$= \frac{\partial}{\partial y}\left(\frac{m^z}{r}\right)\hat{x} - \frac{\partial}{\partial x}\left(\frac{m^z}{r}\right)\hat{y}$$

$$= \left\{ \frac{\partial}{\partial y}(\bar{m}(t-r/c))\frac{1}{r} + \frac{\partial}{\partial y}\left(\frac{1}{r}\right)\bar{m}(t-r/c) \right\}\hat{x}$$

$$- \left\{ \frac{\partial}{\partial x}(\bar{m}(t-r/c))\frac{1}{r} + \frac{\partial}{\partial y}\left(\frac{1}{r}\right)\bar{m}(t-r/c) \right\}\hat{y}$$

only the 1st & 3rd term will give radiation field

$$\rightsquigarrow \frac{\partial}{\partial y}\left(\frac{b}{2c} I_0 \cos\omega(t-r/c)\right)\frac{1}{r}\hat{x}$$

$$= \frac{bI_0}{2cr}\left(-\omega \sin\omega(t-r/c)\right)\left(-\frac{1}{2c}\frac{2y}{r}\right)\hat{x}$$

$$= \frac{bI_0\omega \overset{\sin\omega(t-r/c)}{\cancel{r}} \sin\theta\sin\phi}{2c^2 r^2}\hat{x}$$

$$\rightsquigarrow \frac{\partial}{\partial x}\left(\bar{m}\left(t-\frac{r}{c}\right)\right)\frac{1}{r}\hat{y} = \frac{bI_0\omega \overset{\sin\omega(t-r/c)}{\cancel{r}} \sin\theta\cos\phi}{2c^2 r^2}$$

$$\bar{A} = \frac{bI_0\omega \overset{\sin\omega(t-r/c)}{}}{2c^2r}\sin\theta(\sin\phi\hat{x} - \cos\phi\hat{y})$$

$\hat{x} = \sin\theta\cos\phi\hat{r}$
$+\cos\theta\cos\phi\hat{\theta}$
$-\sin\theta\sin\phi\hat{\phi}$

$$= \frac{bI_0\omega\sin\omega(t-r/c)}{2c^2r}\sin^2\theta(\cos^2\phi-\sin^2\phi)\hat{\phi}$$

$\hat{y} = \sin\theta\sin\phi\hat{r}$
$+\cos\theta\sin\phi\hat{\theta}$
$-\sin\theta\cos\phi\hat{\phi}$

$$\bar{E}_{rad} = -\frac{1}{c}\frac{\partial\bar{A}}{\partial t} = -\frac{bI_0\omega^2\cos\omega(t-r/c)}{2c^3r}\sin^2\theta(\cos^2\phi-\sin^2\phi)\hat{\phi}$$

e) $$\bar{B}_{rad} = \hat{r}\wedge\bar{E}_{rad} = +\frac{bI_0\omega^2\cos\omega(t-r/c)\sin^2\theta(\cos^2\phi-\sin^2\phi)}{2c^3r}\hat{\theta}$$

$$\bar{S} = \frac{c}{4\pi}(\bar{E}_{rad}\times\bar{B}_{rad}) = \frac{c}{4\pi}|E_{rad}|^2\hat{n}$$

$$\frac{dP}{d\Omega} = \bar{S}\cdot\hat{n}\, r^2(1-\bar{\beta}\cdot\hat{r})$$

$$= \frac{c}{4\pi}|E_{rad}|^2r^2 \quad \because \bar{\beta}=0$$

$$\left\langle\frac{dP(r,\theta,\phi)}{d\Omega}\right\rangle = \frac{c}{4\pi}\frac{b^2I_0^2\omega^4}{4c^6r^2}\underbrace{\langle\cos^2\omega(t-r/c)\rangle}_{=\frac{1}{2}}\sin^2\theta(\cos^2\phi-\sin^2\phi)\, r^2$$

$$= \frac{b^2I_0^2\omega^4}{32\pi c^5}\sin^2\theta(1+2\cos^2\phi)$$

$$\rightsquigarrow \frac{\partial}{\partial y}\left(\bar{m}(t-r/c)\right)\frac{1}{r}\hat{x} = \dot{\bar{m}}\left(-\frac{1}{2}\frac{1}{rc}2y\right)\frac{1}{r}\hat{x}$$

$$= -\frac{\dot{\bar{m}}\cdot\hat{x}}{cr}\sin\theta\sin\phi$$

$$\rightsquigarrow \frac{\partial}{\partial x}\left(\bar{m}\left(t-\frac{r}{c}\right)\right)\frac{1}{r}\hat{y} = \dot{\bar{m}}\left(-\frac{1}{2}\frac{1}{cr}2x\right)\frac{1}{r}\hat{y}$$

$$= -\frac{\dot{\bar{m}}\cdot\hat{y}}{cr}\sin\theta\cos\phi$$

$$\bar{A}^{ret}(t,r)\Big|_{rad} = \frac{\dot{\bar{m}}\cdot\hat{y}\sin\theta\cos\phi + \dot{\bar{m}}\cdot\hat{x}\sin\theta\sin\phi}{cr}$$

$$\bar{E}_{rad} = -\frac{1}{c}\frac{\partial A}{\partial t} = -\frac{1}{c^2 r}\left\{\ddot{\bar{m}}\cdot\hat{y}\cos\phi + \ddot{\bar{m}}\cdot\hat{x}\sin\phi\right\}\sin\theta$$

$$\bar{B}_{rad} = \hat{r}\times\bar{E}_{rad} = (\sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z})$$

$$\times\bar{E}_{rad}$$

$$= \nabla\times\bar{A}$$

$$= \begin{pmatrix}\partial/\partial x\\ \partial/\partial y\\ \partial/\partial z\end{pmatrix}\wedge\Big($$