F -09

E&M

Fall 2009

1 Magnetic Materials

Assume the field inside a large piece of magnetic material is $\vec{\mathbf{B}}_0$ so that

$$\vec{\mathbf{H}}_0 = \frac{1}{\mu_0}\vec{\mathbf{B}}_0 - \vec{\mathbf{M}}$$

.

a) Consider a small spherical cavity that is hollowed out of the material. Find the field $\vec{\mathbf{B}}$, at the center of the cavity, in terms $\vec{\mathbf{B}}_0$ and $\vec{\mathbf{M}}$. Also find $\vec{\mathbf{H}}$ at the center of the cavity in terms of $\vec{\mathbf{H}}_0$ and $\vec{\mathbf{M}}$. (3 Points)

b) Do the same calculations for a long needle-shaped cavity running parallel to $\vec{\mathbf{M}}$. (3 Points)

c) Do the same calculations for a thin wafer-shaped cavity perpendicular to $\vec{\mathbf{M}}$. (4 Points)

Hint: Assume the cavities are small enough so that $\vec{\mathbf{M}}$, $\vec{\mathbf{B}}_0$ and $\vec{\mathbf{H}}_0$ are essentially constant. The field of a magnetized sphere is $\vec{\mathbf{B}} = \frac{2}{3}\mu_0\vec{\mathbf{M}}$ and the field inside a long solenoid is $\mu_0 K$ where K is the surface current density.

$\bar{B} = \frac{8\pi}{3} \cdot \bar{M}$

Prob 1 (Gaussian)

$$\bar{H}_0 = \bar{B}_0 - 4\pi \bar{M} \Rightarrow \bar{B}_0 = \bar{H}_0 + 4\pi\bar{M}$$

a) The field of a magnetized sphere $\bar{B}_0 = \frac{8\pi}{3}\bar{M}$

We can consider, the spherical cavity as a sphere of opposite magnetization

$$\therefore \bar{B}' = -\frac{8\pi}{3}\bar{M} \Rightarrow \bar{H}' = \bar{B}' + 4\pi\bar{M} = \frac{4\pi}{3}\bar{M}$$

Thus, $\rightarrow \bar{B} = \bar{B}_0 - \frac{8\pi}{3}\bar{M}$

$$\rightarrow \bar{H} = H_0 + \frac{4\pi}{3} M$$

b) We can consider the needle shaped cavity as a long solenoid

The field inside a solenoid $\bar{B} = \frac{4\pi}{c}\vec{K}$

but $\bar{K}_b = c\,\bar{M} \wedge \hat{n}$

$(-\hat{z}) \wedge \hat{r}$

$$\Rightarrow \bar{B} = 4\pi\,\bar{M} \wedge \hat{n}$$

if the magnetization is $(-\bar{M})$ then the field of the needle $\quad \bar{B}' = -4\pi \bar{M} \wedge \hat{n} = -4\pi M \hat{\phi} = -4\pi \bar{M}$

$\rightsquigarrow \quad \bar{B} = \bar{B}_0 - 4\pi \bar{M}$

and $\quad \bar{H}' = \bar{B}' + 4\pi \bar{M} = 0$

$\rightsquigarrow \quad \bar{H} = \bar{H}_0$

c) For thin wafer the bound current is very small

So, $K_b = 0$

$\bar{B}' = 0 \qquad \bar{H}' = +4\pi \bar{M}$

$\bar{B} = B_0 - \bar{B}' = B_0$

$\bar{H} = \bar{H}_0 + \bar{H}' = \bar{H}_0 + 4\pi \bar{M}$

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2 Space-charge-limited Thermionic Planar Diode

Consider a planar diode with a grounded, hot metallic cathode at $x = 0$ and a metallic anode plate at $x = H$, which is held at an electrical potential of V_p relative to ground. [Cathode and anode plates are infinite in the y and z directions.] The cathode is very hot and emits copious electrons such that the diode is "space charge limited", that is: the electric field **at the cathode is zero**. The current density J is constant and in the -x direction. [Ignore any transient effects.]

In this problem let : $V(x)$ be the electric potential, $E(x)$ be the electric field, s(x) be the velocity of an electron, $\rho(x)$ be the charge density, and m and -e be the mass and charge of an electron respectively.

a) State whether you are using MKS or cgs units. (1 Point)

b) Find $\rho(x)$ as a function of $V(x)$ and any other relevant variables. (2 Points)

c) Use Poisson's equation to find the differential equation for $V(x)$. (2 Points)

d) State the boundary conditions for $E(x)$ at $x = 0$ and $V(x)$, at $x = 0$ and $x = H$. (2 Points)

Work e) or f) on a separate sheet of paper and submit only the one you wish to be graded.

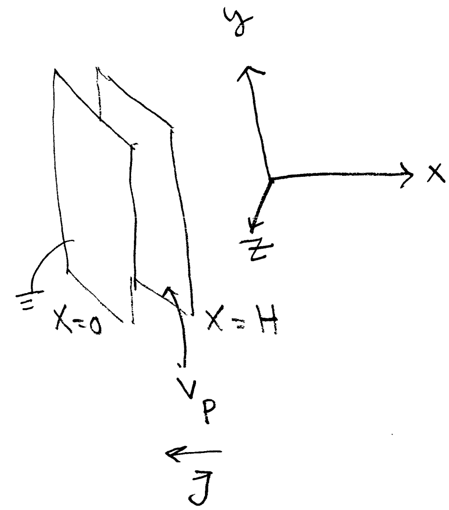
e) Solve for V(x) in terms of V_p and H using results of c) and d). ((3 Points for part e or f)

Hint: multiply both sides of your differential equation by $dV(x)/dx$ and recall that: $(dV/dx)(d^2V/dx^2) = \frac{1}{2}d(dV/dx)^2/dx$

If you have trouble using the above hint to complete part e), then try f).

f) Assume $V(x)$ is of the form: $Ax^n + Bx + C$ and solve for $V(x)$ in terms of V_p and H using parts c) and d) above. Find the current density J in terms of V_p and H. (3 Points for part e or f)

Prob 2 (Gaussian)



a) Gaussian

b)

$$dq = \rho(x) A\, dx$$

$$\Rightarrow \quad I = \frac{dq}{dt} = \rho(x) A \frac{dx}{dt} = \rho(x) A\, s(x)$$

From conservation of energy,

$$\frac{1}{2} m s^2(x) = qV = -qV(x)$$

$$\Rightarrow \quad s(x) = \sqrt{\frac{2qV(x)}{m}}$$

$$\Rightarrow \quad I = \rho(x) A \sqrt{\frac{2qV(x)}{m}}$$

$$\Rightarrow \quad \rho(x) = \frac{I}{A} \sqrt{\frac{m}{2eV(x)}}$$

$$\Rightarrow \quad \rho(x) = J \sqrt{\frac{m}{2eV(x)}}$$

c) $$\nabla^2 V(x) = -4\pi \rho(x)$$

$$\Rightarrow \frac{d^2 V(x)}{dx^2} = \underbrace{-4\pi J \sqrt{\frac{m}{2e}}}_{\beta} V(x)^{-\frac{1}{2}}$$

$$\Rightarrow \frac{d^2 V(x)}{dx^2} = \beta V(x)^{-\frac{1}{2}}$$

d) $E(0) = 0$ $\qquad$ $E(x=H) = \frac{V(H)}{H}$

$V(0) = 0$ $\qquad$ $V(x=H) = V_P$

e) $$\frac{d^2 V(x)}{dx^2} = \beta V(x)^{-\frac{1}{2}}$$

$$\Rightarrow \frac{dV(x)}{dx}\frac{d^2 V(x)}{dx^2} = \beta \frac{dV(x)}{dx} V(x)^{-\frac{1}{2}}$$

$$\Rightarrow \frac{d\left(\frac{dV}{dx}\right)^2}{dx} = \left(2\beta V(x)^{-\frac{1}{2}}\right)\frac{dV(x)}{dx}$$

$$\Rightarrow \int_0^x d\left(\frac{dV}{dx}\right)^2 = 2\beta \int_0^{V(x)} V(x)^{-\frac{1}{2}} dV(x)$$

$$\Rightarrow \left(\frac{dV(x)}{dx}\right)^2 = 2\beta \; 2V(x)^{\frac{1}{2}} \Big|_0^{V(x)}$$

$$\Rightarrow \left(\frac{dV(x)}{dx}\right)^2 = 4\beta V(x)^{1/2}$$

$$\Rightarrow V(x)^{-\frac{1}{4}} dV(x) = 2\sqrt{\beta}\, dx$$

$$\Rightarrow 2\sqrt{\beta}\int dx = \int V(x)^{-\frac{1}{4}} dV(x)$$

$$\Rightarrow 2\sqrt{\beta}\, x = \frac{4}{3} V(x)^{3/4} + \text{const}$$

At $x = H$, $V(x) = V_p$

$$\Rightarrow 2\sqrt{\beta} H = \frac{4}{3} V_p^{3/4} + \text{const.}$$

$$\Rightarrow \text{const} = 2\sqrt{\beta} H - \frac{4}{3} V_p^{3/4}$$

$$\Rightarrow \frac{4}{3} V(x)^{3/4} = 2\sqrt{\beta}\, x - 2\sqrt{\beta} H + \frac{4}{3} V_p^{3/4}$$

$$\Rightarrow V(x)^{3/4} = \frac{3}{2}\sqrt{\beta}(x - H) + V_p^{3/4}$$

$$\Rightarrow V(x) = \left\{\frac{3}{2}\sqrt{\beta}\,(x-H) + V_p^{3/4}\right\}^{4/3}$$

but $\sqrt{\beta}$ is imag!

3 Wire

An infinitely-long, thin wire (radius b) is coated with a dielectric (relative dielectric constant $k = \epsilon/\epsilon_0$ with radius $a > b$). The metal wire has charge per unit length λ

a) Find the electric displacement $\vec{\mathbf{D}}$ everywhere. (2 points)

b) Find the electric field $\vec{\mathbf{E}}$ everywhere. (2 points)

c) Find the polarization $\vec{\mathbf{P}}$ everywhere. (3 points)

d) Find **all** the bound charge everywhere. (3 points)

Prob 3 (Gaussian)

a) $\underline{\underline{r > b}}$

$$\int \bar{D} \cdot d\bar{a} = 4\pi Q_{f\,enc}$$

$$\bar{D}\,(2\pi r L) = 4\pi \lambda L$$

$$\Rightarrow \quad \bar{D} = \frac{2\lambda}{r}\,\hat{r}$$

b) $\underline{\underline{b < r < a}}$

$$\bar{E} = \frac{2\lambda}{\epsilon r}\,\hat{r}$$

$\underline{r > a}$

$$E = \frac{2\lambda}{r}\,\hat{r}$$

c) $\bar{D} = \bar{E} + 4\pi \bar{P}$

$\underline{\underline{a > r > b}}$

$$\bar{P} = \frac{1}{4\pi}(\bar{D} - \bar{E}) = \frac{\lambda}{2\pi r}\left(\frac{1}{\epsilon} - 1\right)\hat{r}$$

$\underline{\underline{r > a}}$

$$P = 0$$

d) $\sigma_p = \bar{P} \cdot \hat{n}$

for $r = b$; inner surface, $\hat{\eta} = -\hat{r}$

for $r = a$; outer surface, $\hat{\eta} = \hat{r}$

$$\sigma_p = -\frac{\lambda}{2\pi b}\left(\frac{1}{\epsilon} - 1\right)$$

$$= +\frac{\lambda}{2\pi a}\left(\frac{1}{\epsilon} - 1\right)$$

$$\rho_b = -\bar{\nabla} \cdot \bar{P} = -\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\lambda}{2\pi r}\right)\left(\frac{1}{\epsilon} - 1\right) = 0$$

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4 Electromagnetic Waves

Consider a plane electromagnetic wave with propagation vector $\vec{\mathbf{k}}$ and angular frequency ω. Construct the four-vector $k^\mu = (\omega/c, \vec{\mathbf{k}})$. Use the metric $g_{\mu\nu} = diag(-1, 1, 1, 1)$

a)Verify that $k_\mu k^\mu = 0$. (2 points)

b) In terms of the position four-vector $x^\mu = (ct, \vec{\mathbf{r}})$, show that the plane wave propagation factor is

$$e^{ik_\mu x^\mu} = e^{i(\vec{\mathbf{k}}\cdot\vec{\mathbf{r}} - \omega t)}.$$

(2 points)

c) Now use Lorentz transformations to show that radiation of frequency ω propagating at an angle θ with respect to the z-axis, will, to an observer moving with relative velocity $\nu = \beta c$ along the z axis, have the frequency

$$\omega' = \frac{1}{\sqrt{1-\beta^2}}\omega(1 - \beta\cos\theta).$$

(2 points)

d) Further show that the moving observer sees the radiation propagating at an angle θ' with repect to the z-axis, where

$$\cos\theta' = \frac{\cos\theta - \beta}{1 - \beta\cos\theta},$$

which is aberration. (2 points)

e) Find θ' explicitly if $|\beta| << 1$. (2 points)

4. (Gaussian)

a)

$$K^\mu = \begin{pmatrix} \frac{\omega}{c} \\ \vec{K} \end{pmatrix} \qquad g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$K^\mu K_\mu = K^\mu g_{\mu\nu} K^\nu$$

$$= \left(\frac{\omega}{c} \;\; \vec{K}\right) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \left(\frac{\omega}{c} \;\; \vec{K}\right)$$

$$= \left(\frac{\omega}{c} \;\; \vec{K}\right) \begin{pmatrix} -\frac{\omega}{c} \\ \vec{K} \end{pmatrix}$$

$$\equiv -\frac{\omega^2}{c^2} + K^2 = -\frac{\omega^2}{c^2} + \frac{\omega^2}{c^2} = 0$$

b)

$$K_\mu x^\mu = K_\mu g^{\mu\nu} x_\nu$$

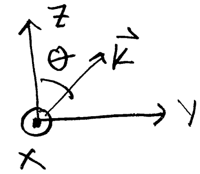
$$= \left(-\frac{\omega}{c} \;\; \vec{K}\right) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -ct \\ \bar{r} \end{pmatrix}$$

$$= \left(-\frac{\omega}{c} \;\; \bar{K}\right) \begin{pmatrix} +ct \\ \bar{r} \end{pmatrix}$$

$$= -\omega t + \bar{K} \cdot \bar{r}$$

$$= \bar{K} \cdot \bar{r} - \omega t$$

c)



$$\begin{pmatrix} \frac{\omega'}{c} \\ k_x' \\ k_y' \\ k_z' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} \frac{\omega}{c} \\ 0 \\ k\sin\theta \\ k\cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \gamma\frac{\omega}{c} - \gamma\beta k\cos\theta \\ 0 \\ k\sin\theta \\ -\gamma\beta\frac{\omega}{c} + \gamma k\cos\theta \end{pmatrix}$$

$$\frac{\omega'}{c} = \gamma\frac{\omega}{c} - \gamma\beta\frac{\omega}{c}\cos\theta$$

$$= \gamma\frac{\omega}{c}\left(1-\beta\cos\theta\right)$$

$$\Rightarrow \quad \omega' = \frac{\omega}{\sqrt{1-\beta^2}}\left(1-\beta\cos\theta\right)$$

d) $$k_y' = k\sin\theta$$

$$k_z' = -\gamma\beta\frac{\omega}{c} + \gamma\frac{\omega}{c}\cos\theta$$

$$\frac{\omega'}{c}\cos\theta' = \gamma\frac{\omega}{c}(\cos\theta-\beta)$$

$$\Rightarrow \gamma\frac{\omega}{c}(1-\beta\cos\theta)\cos\theta' = \gamma\frac{\omega}{c}(\cos\theta-\beta)$$

$$\Rightarrow \cos\theta' = \frac{\cos\theta-\beta}{1-\beta\cos\theta}$$

e) $$\theta' = \cos^{-1}\left\{\frac{\cos\theta-\beta}{1-\beta\cos\theta}\right\}$$

if $|\beta| \ll 1$

$$\theta' = \theta(0) + \dot{\theta}(0)\cdot\beta \Rightarrow \text{taylor expansion}$$

Find $\dot{\theta}(0) = \sin\theta$

$$\theta' = \theta + \beta\sin\theta.$$

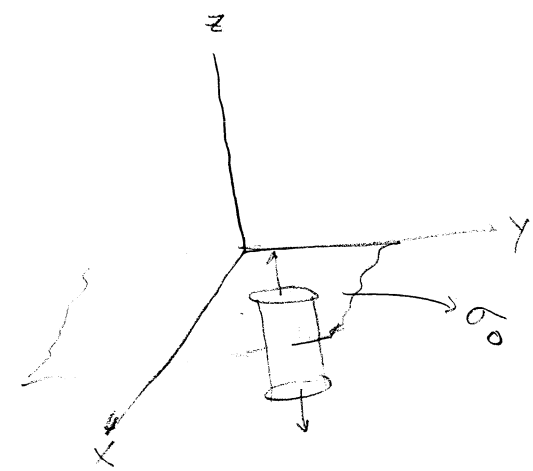
5 Thin Infinite Sheet

a) Compute the 4-current $J^\alpha(x^\beta)$ and the E&M fields for a stationary, thin, and infinite sheet of charge located at $z = 0$ in the lab. Assume the surface charge density is a constant σ_0. (4 points)

b) Now assume you move with speed $v < c$ in the x-direction relative to the lab. What is the 4-current $J'^\alpha(x^\beta)$ and E&M field in your frame? (6 points)

Prob 5 (Gaussian)

a) $$J'^{\alpha}(x'^{\beta}) = \begin{pmatrix} c\rho' \\ \bar{J}' \end{pmatrix} = \begin{pmatrix} c\sigma_0 \\ 0 \end{pmatrix}$$



$$\int \bar{E}\cdot d\bar{a} = 4\pi \int \sigma\, d^2x$$

$$\int \bar{E}\cdot d\bar{a} + \int \bar{E}\cdot d\bar{a} = 4\pi \int_0^{r'} \sigma_0\, r' dr' d\theta'$$

$$2E\pi r'^2 = 4\pi\sigma_0 \cdot 2\pi \frac{r'^2}{2}$$

$$\Rightarrow \quad \bar{E}'(\bar{r}') = 2\pi\sigma_0\, \hat{z}$$

$$\Rightarrow \quad \bar{B}'(\bar{r}') = 0$$

b) $$J^{\beta}(x^{\beta}) = \begin{pmatrix} c\rho \\ \bar{J} \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} c\sigma_0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma c\sigma_0 \\ \gamma\beta c\sigma_0 \end{pmatrix}$$

$$\bar{J} = \gamma\beta c\sigma_0\, \hat{x}$$

$$\bar{E} = \bar{E}'_{\parallel} + \gamma\left(\bar{E}'_{\perp} - \bar{\beta}\wedge\bar{B}'\right)$$

$$\bar{B} = \bar{B}'_{\parallel} + \gamma\left(\bar{B}'_{\perp} + \bar{\beta}\wedge\bar{E}'\right)$$

$$\bar{E} = \gamma 2\pi \sigma_0 \hat{z}$$

$$\bar{B} = \bar{\beta} \wedge \bar{E} = \gamma \beta 2\pi \sigma_0 (-\hat{y})$$

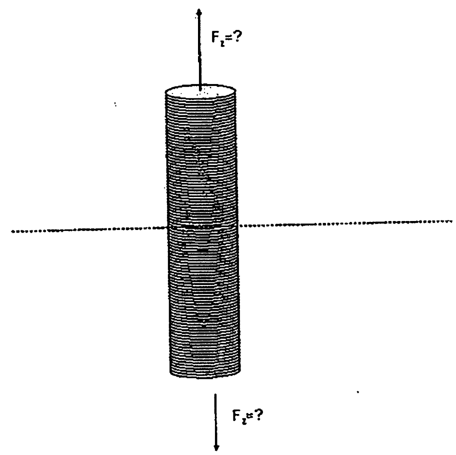


Figure 1: Stack of Disks for Stress Tensor Problem. Problem 6

6 Stress Tensor

Consider a long cylinder of radius a and length L made up of a stack of infinitesimally thin discs (See Figure). Assume the disks alternate between disks with charge density ρ and angular velocity $\omega\hat{z}$ and disks with charge density $-\rho$ and angular velocity $-\omega\hat{z}$.

a) Specify the system of units you will be using. (1 points)

b) write down an expression fo the charge and current density in any small volume (of dimension larger than the infinitesimal thickness of the disks). (1 points)

b) Find the electromagnetic field everywhere. (2 points)

c) Find the Maxwell Stress Tensor everywhere. (2 points)

d) Use your answer to part c to find the force of the top half of the cylinder on the bottom half. (2 points)

e) Is the force attractive or repulsive? (2 points)

Prob 6 (Gaussian)

a) Gaussian

b) $\rho_+ = \rho$ $\quad \boxed{\rho = 0}$

$\rho_- = -\rho$

$v = \omega \times r = \omega r = \omega a$

$J_+ = \rho\omega r(\hat{z} \wedge \hat{r}) = \rho\omega r\hat{\phi}$

$J_- = -\rho\omega r(-\hat{z} \wedge \hat{r}) = \rho\omega r\hat{\phi}$

Let, there are N disks in total

so thickness of each disk $\Rightarrow \Delta d = \frac{N}{L}$

if

In a vol. of $\pi a^2 \Delta l = \pi a^2 (\alpha+\beta)\Delta d \Rightarrow q = (\alpha\rho - \beta\rho)\pi a^2 (\alpha+\beta)\Delta d$

$$\rho_{net} = \frac{q}{\pi a^2 \Delta d(\alpha+\beta)}$$

$$= \frac{\rho(\alpha-\beta)\pi a^2\Delta d(\alpha+\beta)}{\pi a^2(\alpha+\beta)\Delta d}$$

$$= \rho(\alpha-\beta)$$

$J_{net} = \rho\omega a(\alpha+\beta)\hat{\phi}$

b) Using Gauss. law

$\underline{r < a}$

$$\int \bar{E} \cdot d\bar{a} = 4\pi \int \rho_{net} \, dv$$

$$\Rightarrow E \, (2\pi r) \, \Delta l = 4\pi \rho_{net} \, \pi r^2 \Delta l$$

$$\Rightarrow \bar{E} = 2\pi r \, \rho(\alpha - \beta) \, \hat{r}$$