F-07

Electrodynamics Qualifier Examination

August 15, 2007

General Instructions: In all cases, be sure to state your system of units. Show all your work, write only on one side of the designated paper, and if you get stuck on one part, assume a result and proceed onward. The points given for each part of each problem are indicated. Each problem carries equal weight.

1. A point charge Q is located a distance d from the center of a grounded sphere of radius R, as shown in the figure. The point charge is located outside the sphere, that is, $d > R$. Use the image method to answer the following questions.

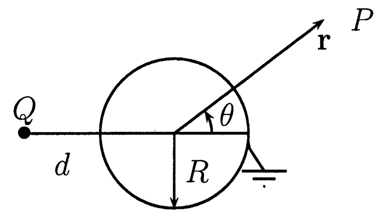


Figure 1: Point charge Q exterior to a grounded conducting sphere.

a) 2 pts. Find the position and magnitude of the image charge Q' that will make the potential zero on the surface of the sphere.

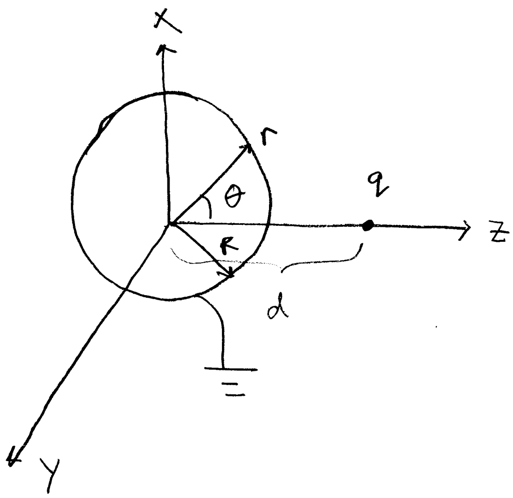
b) 2 pts. Show that the image method is applicable to this problem by proving that the result in a) will make the potential zero at an arbitrary point on the surface of the sphere.

c) 2 pts. Write down the expression for the potential at an arbitrary point $P(r, \theta)$ outside the sphere. Take the origin of the coordinate system to be the center of the sphere.

d) 2 pts. Use the result of part c) to calculate the radial component of the electric field, E_r, outside the sphere.

e) 2 pts. Use Gauss' law to find the total induced charge on the surface of the sphere.

Prob 1 (Gaussian)



a) Suppose, the image charge Q' is at $z'\hat{z}$

$$\Phi_q = \frac{q}{|\bar{r} - d\hat{z}|} = \frac{q}{(r^2 + d^2 - 2rd\cos\theta)^{1/2}}$$

$$\Phi_{Q'} = \frac{Q'}{|\bar{r} - z'\hat{z}|} = \frac{Q'}{(r^2 + z'^2 - 2rz'\cos\theta)^{1/2}}$$

$$\Phi_q(r=R) + \Phi_{Q'}(r=R) = 0$$

$$\Rightarrow \frac{q}{(R^2 + d^2 - 2Rd\cos\theta)^{1/2}} = \frac{Q'}{(R^2 + z'^2 - 2Rz'\cos\theta)^{1/2}}$$

$$\Rightarrow q^2(R^2 + z'^2 - 2Rz'\cos\theta) = Q'^2(R^2 + d^2 - 2Rd\cos\theta)$$

$$\Rightarrow q^2(R^2 + z^2) - 2q^2Rz'\cos\theta = Q'^2(R^2 + d^2) - 2Q'^2Rd\cos\theta$$

Equating the co-efficients

$$\left.\begin{aligned} &\Rightarrow q^2(R^2 + z'^2) = Q'^2(R^2 + d^2) \\ &\Rightarrow q^2 z' = Q'^2 d \end{aligned}\right\}$$

$$\Rightarrow z' = \frac{Q'^2}{q^2} d$$

$$\Rightarrow q^2 (R^2 + z'^2) = Q'^2 (R^2 + d^2)$$

$$\Rightarrow \frac{q^2}{Q'^2} R^2 + \frac{q^2}{Q'^2} \frac{Q'^4}{q^4} d^2 = R^2 + d^2$$

$$\Rightarrow \frac{q^2}{Q'^2} R^2 + \frac{Q'^2}{q^2} d^2 = R^2 + d^2$$

$$\Rightarrow R^2 \left(1 - \frac{q^2}{Q'^2}\right) = \left(\frac{Q'^2}{q^2} - 1\right) d^2$$

$$\Rightarrow R^2 \left(\frac{Q'^2 - q^2}{Q'^2}\right) = + \frac{Q'^2 - q^2}{q^2} d^2$$

$$\Rightarrow \frac{R^2}{Q'^2} = \frac{d^2}{q^2}$$

$$\Rightarrow \frac{R^2}{d^2} q^2 = Q'^2 \Rightarrow \frac{Q'^2}{q^2} = \frac{R^2}{d^2}$$

$$\Rightarrow \boxed{z' = \frac{R^2}{d}}$$

$$\Rightarrow q^2\left(R^2+\frac{R^4}{d^2}\right) = Q'^2(R^2+d^2)$$

$$\Rightarrow q^2R^2\left(1+\frac{R^2}{d^2}\right) = Q'^2 d^2\left(1+\frac{R^2}{d^2}\right)$$

$$\Rightarrow Q'^2 = \frac{q^2R^2}{d^2}$$

$$\Rightarrow Q' = \pm\left(\frac{R}{d}\right)q$$

→ don't understand the pos. soln

Image charge, $\boxed{Q' = -\left(\frac{R}{d}\right)q}$

b) $\Phi_q(r=R) + \Phi_{Q'}(r=R)$

$$= \frac{q}{(R^2+d^2-2Rd\cos\theta)^{1/2}} + \underbrace{\frac{-\left(\frac{R}{d}\right)q}{\left(R^2+\frac{R^4}{d^2}-2\frac{R^3}{d}\cos\theta\right)^{1/2}}}$$

$$+ \frac{-(R/d)q}{\left(\frac{R}{d}\right)\left(d^2+R^2-2Rd\cos\theta\right)^{1/2}}$$

$$= 0$$

c)

$$\Phi(r,\theta) = \frac{q}{\sqrt{r^2+d^2-2rd\cos\theta}} + \frac{\left(-\frac{R}{d}\right)q}{\sqrt{r^2+\frac{R^4}{d^2}-2r\frac{R^2}{d}\cos\theta}}$$

d)

$$E_r = -\frac{\partial \Phi(r,\theta)}{\partial r}$$

$$= -\frac{\left(-\frac{1}{2}q\right)(2r-2d\cos\theta)}{(r^2+d^2-2rd\cos\theta)^{3/2}} - \frac{\left(-\frac{1}{2}\right)\left(-\frac{R}{d}\right)q\left(2r-2\frac{R^2}{d}\cos\theta\right)}{\left(r^2+\frac{R^4}{d^2}-2r\frac{R^2}{d}\cos\theta\right)^{3/2}}$$

$$= \frac{(r-d\cos\theta)q}{(r^2+d^2-2rd\cos\theta)^{3/2}} - \frac{q\left(\frac{R}{d}\right)\left(r-\frac{R^2}{d}\cos\theta\right)}{\left(r^2+\frac{R^4}{d^2}-2r\frac{R^2}{d}\cos\theta\right)^{3/2}}$$

e)

$$\nabla\cdot E = 4\pi\sigma$$

$$\Rightarrow \sigma = \frac{1}{4\pi}\left.\frac{\partial E_r}{\partial r}\right|_{r=R}$$

$$= \frac{1}{4\pi}\frac{\partial}{\partial r}\Big\{$$

2. A thin dielectric film of thickness δ and index of refraction n_2 lies between dielectric media of indices of refraction n_1 and n_3 as shown in the figure. Assume $n_1 < n_2 < n_3$. A light wave of amplitude E_I,

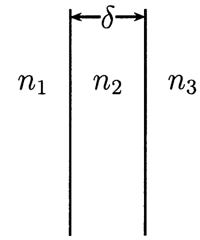


Figure 2: Thin dielectric film of thickness δ and index of refraction n_2 separating media of indices of refraction n_1 and n_3, respectively.

wavenumber k_I, and frequency ω is incident normally from the left. In this problem we will find the conditions for there to be no reflected wave. This phenomenon is of great utility, for example, in reducing light losses in optical equipment with many glass surfaces.

a) 3 pts. First assume a single interface, at $x = 0$, bounding two parallel semi-infinite dielectrics of indices of refraction n_1 and n_2, respectively. Assuming a normally incident plane wave in medium 1, calculate the transmission and reflection coefficients, t_{12} and r_{12}, defined by

$$E(x) = E_I \left(e^{ik_1 x} + r_{12}\, e^{-ik_1 x}\right), \quad x < 0,$$
$$E(x) = E_I\, t_{12}\, e^{ik_2 x}, \quad x > 0,$$

where k_i is the propagation constant in the medium i.

b) 3 pts. Using the reflection and transmission coefficients for each interface r_{ij} and t_{ij} found in part a) for normal incidence, namely,

$$r_{ij} = \frac{n_i - n_j}{n_i + n_j}, \quad t_{ij} = \frac{2n_i}{n_i + n_j},$$

write the ratio of reflected to incident wave amplitudes, E_R/E_I, for the three medium interface being considered, as an infinite series taking multiple reflections into account. (E_R is the reflected wave amplitude in medium 1.)

c) 1 pt. Show that the series in part a) can be written in closed form using

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots.$$

d) 3 pts. Now show that with $n_2 = \sqrt{n_1 n_3}$ and $\delta = N\lambda_2/4$ (N is an odd integer and λ_2 is the wavelength in the film), $E_R/E_I = 0$.

3. Two parallel-plate capacitors are constructed of square plates with area A. In capacitor 1, which contains no dielectric material, the plates are separated by a distance d. Capacitor 2 has a slab of dielectric material with permittivity $\varepsilon = K\varepsilon_0$, where ε_0 is the permittivity of free space ($= 1$ in Gaussian units). The dielectric slab has area A and thickness H. The separation between the two parallel plates of capacitor 2 is $H(1+1/K)$. (You may neglect fringe fields at the edges of the plates.)

   The top plates of each capacitor are suspended by insulating threads at opposite ends of the beam of an equal-arm balance. The top plates have equal masses, and the beam is initially balanced and locked into place with each top plate parallel to its corresponding bottom plate which is supported by an insulator. See Figure.

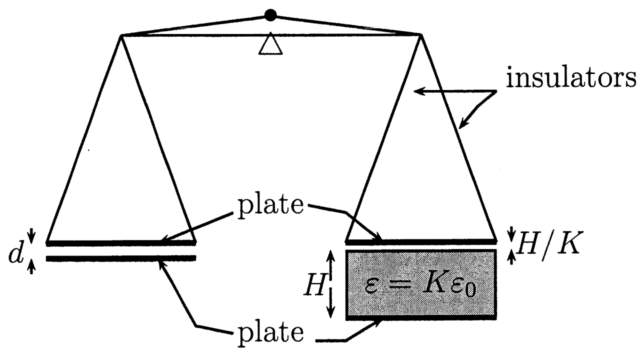


Figure 3: Parallel-plate capacitors mounted on balance.

a) 2 pts. Calculate C_1 and C_2, the capacitances of the two capacitors.

b) 2 pts. Each capacitor is charged by connecting it to a battery of voltage V_0, which is subsequently disconnected. If the energy stored in each capacitor is identical, find d in terms of A, H, and K, and determine the charges Q_1 and Q_2 on capacitor 1 and 2 respectively.

c) 4 pts. Now the beam is unlocked (without discharging the capacitors) and displaced slightly so that the separation between the plates of capacitor 1 is now equal to $d + x$ and the separation between the plates of capacitor 2 is now equal to $H(1 + 1/K) - x$. (Assume that $x \ll H/K$, and that the charge on each capacitor remains the same as in part b).) Calculate the new capacitance of C_1 and C_2 and the total energy U_{total} stored on the system of two capacitors.

Is the system stable (tending to decrease x), unstable (tending to increase x) or neutral (tending to keep x constant)?

d) 2 pts. If in part c), the two top plates are connected together with conducting wires of negligible mass, and likewise the two bottom plates are electrically connected to each other, is the system stable, unstable, or neutral? Support your answer with a physical argument.

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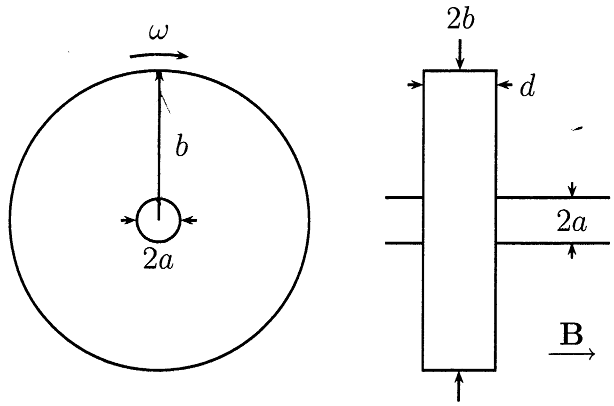

Figure 4: End (left) and side (right) views of the Faraday disc generator. The disc, of radius b, rotates on a shaft of radius a, at constant angular frequency ω. The thickness of the disc is d. A constant magnetic field $\mathbf{B}$ is applied in the direction of the shaft. The shaft and the disc are both conductors, and are in contact with an external resistor R connected between brushes making contact with the shaft and the circumference of the disc.

4. Consider the Faraday disk generator, shown in the Figure, which is a fairly realistic design as might be used in mining and refining. A copper disk, with conductivity σ and inner radius a, outer radius b, and thickness d, is rotated at a constant angular speed ω in a constant magnetic field B parallel to the axis of rotation, that is, perpendicular to the disk. There are continuous radial contacts with the disk at $s = a$ and $s = b$, where s is the radial distance from the center of the disk. Current flows through an external resistor R.

a) 3 pts. What is the emf between the contacts?

b) 3 pts. There must be an electric field associated with this emf due to the internal resistance of the copper. Use this to find the radial current $I(s)$ in the disk as a function of the distance from the axis of the disk.

c) 4 pts. Use the result of part b) to determine the charge density ρ in the disk in terms of ω, s, B, a, b, d, σ, and R.

$I = \frac{\mathcal{E}}{R}$

Prob 4 (Gaussian)

a) $\mathcal{E} = -\frac{1}{c}\frac{d}{dt}\int \bar{B}\cdot d\bar{a}$

$\bar{f} = \frac{\bar{v}}{c} \wedge \bar{B} = \frac{B\omega r}{c}\hat{r}$ $\quad \bar{v} = \omega r\hat{\phi}$, $\bar{B} = B\hat{z}$

$\mathcal{E} = \int \bar{f}\cdot d\bar{l}$

$\Rightarrow \mathcal{E} = \int_a^b \frac{B\omega}{c} r\, dr$

$\Rightarrow \mathcal{E} = \frac{B\omega}{2c}(b^2 - a^2)$

b) $I(r) = \frac{\mathcal{E}}{R + r}$

where, $r = \frac{1}{\sigma}\int_a^b \frac{dr}{2\pi r d} = \frac{1}{2\pi d}\ln\left(\frac{b}{a}\right)$

$I(r) = \frac{\mathcal{E}}{R + \frac{\ln(b/a)}{2\pi d}} = \frac{B\omega(b^2-a^2)/2c}{R + \frac{\ln(b/a)}{2\pi d}}$

c) $\oint \bar{E}\cdot d\bar{l} = \mathcal{E}$

$\Rightarrow E\, 2\pi r = \frac{B\omega}{2c}(b^2 - a^2)$

$\Rightarrow \bar{E} = \frac{B\omega}{4\pi cr}(b^2 - a^2)$

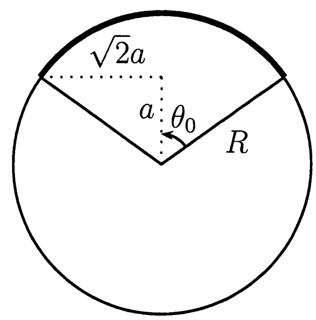

Figure 5: A solid conducting sphere having a cone cut out of its apex. The cone extends to the center of the sphere. The half angle of the cone is $\theta_0 = \arctan\sqrt{2}$. The cone is covered with a metal foil which is insulated from the rest of the sphere. (Thus the interior of the cone is hollow.) A charge Q is placed on the metal foil.

5. There is a solid metal sphere of radius R with a cone cut out of it, of half-angle θ_0, $\tan\theta_0 = \sqrt{2}$, as shown in the Figure. The cone is covered with a thin spherical cap to complete the outer surface of the sphere. This spherical cap, made of metal foil, is insulated from the rest of the sphere. A total charge Q is placed on this foil cap. Find the potential everywhere, as follows. Use polar coordinates centered at the center of the sphere, and with the cone axis coinciding with the z axis.

a) 2 pts. Write down the form of the Legendre polynomial expansion for the potential in each region, $r > R$, $r < R$ with $\theta > \theta_0$, and $r < R$ with $\theta < \theta_0$.

b) 2 pts. Determine the potential everywhere in the solid metal region, $r < R$, $\theta > \theta_0$ in terms of the total charge on the system.

c) 6 pts. Determine the potential inside the cone ($r < R$, $\theta < \theta_0$) and outside the sphere ($r > R$) by requiring that the discontinuity in the radial component of the electric field be proportional to the charge density on the foil, while the potential must be continuous across the foil.

6. The space-time coordinates of a particle are given by the four-vector $x^\mu = (ct, \mathbf{x})$. Here let us use the metric $g_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$, so the corresponding covariant vector is $x_\mu = (-ct, \mathbf{x})$. The four-dimensional gradient operator is $\partial_\mu = \partial/\partial x^\mu$.

a) 3 pts. The relativistic equation of motion of a particle with rest mass m_0 and charge e is in Gaussian or Heaviside-Lorentz units

$$m_0 \frac{d^2 x^\mu}{d\tau^2} = \frac{e}{c} F^{\mu\nu} \frac{dx_\nu}{d\tau},$$

where the proper-time interval is related to the coordinate time interval by $d\tau = \sqrt{1-\beta^2}\, dt$, $\beta = v/c$. Here $F^{\mu\nu} = -F^{\nu\mu}$ is the field strength tensor. By requiring that the spatial components of this equation agree with the Lorentz force law, $\mathbf{F} = e\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right)$, determine the nonzero components of $F^{\mu\nu}$ in terms of $\mathbf{E}$ and $\mathbf{B}$.

b) 3 pts. Show that the time component of the relativistic equation of motion in part b) is the equation of energy conservation.

c) 4 pts. Show that the particle stress tensor,

$$T^{\mu\nu}(x) = \int_{-\infty}^{\infty} d\tau\, m_0 c \frac{dx^\mu(\tau)}{d\tau} \frac{dx^\nu(\tau)}{d\tau} \delta^{(4)}(x - x(\tau)),$$

where $x^\mu(\tau)$ is the space-time trajectory of the particle, satisfies

$$\partial_\nu T^{\mu\nu} = \frac{1}{c} F^{\mu\nu} j_\nu.$$

Here the electric current is given by a similar proper-time integral,

$$\frac{1}{c} j^\mu(x) = \int_{-\infty}^{\infty} d\tau\, e \frac{dx^\mu}{d\tau} \delta^{(4)}(x - x(\tau)).$$

This is an expression of energy-momentum conservation, stating the balance between the particle and field contributions.

Prob 6 (Gaussian)

$$m_0 \frac{d^2x^\mu}{d\tau^2} = \frac{e}{c} F^{\mu\nu} \frac{dx_\nu}{d\tau}$$

$\Rightarrow$ $\underline{\mu = i, \nu = 0}$

In rest frame, $d\tau = dt$

$$m_0 \frac{d^2x^i}{d\tau^2} = \frac{e}{c} F^{i0} \frac{dx_0}{d\tau} \stackrel{!}{=} e(\bar{E} + \frac{\bar{v}}{c} \wedge \bar{B}) = \bar{F}$$

$$\Rightarrow m_0 \frac{d^2x}{dt^2} = \frac{e}{c} F^{i0} \frac{d(ct)}{dt} = -e F^{i0}$$

$$\Rightarrow F^{i0} = -\bar{E} = +F^{0i}$$

$\Rightarrow$ $\underline{\mu = i, \nu = j}$

$$m \frac{d^2x^i}{dt^2} = \frac{e}{c} F^{ij} \frac{dx_j}{dt} = e\bar{E} + \frac{e}{c}(\bar{v} \wedge \bar{B})$$

$$+\frac{e}{c} v^i \wedge B^j = \frac{e}{c} \epsilon_{ijk} v^j B^k$$

$$= \frac{e}{c} F^{ij}(-v^j)$$

$$\Rightarrow F^{ij} = -\epsilon_{ijk} B^k$$

F^{ij} (F^{00} F^{10} F

b) $m_0 \frac{d^2 x^\mu}{d\tau} = \frac{e}{c} F^{\mu\nu} \frac{dx_\nu}{d\tau}$

$$\Rightarrow \frac{dp^\mu}{d\tau} = \frac{e}{c} F^{\mu\nu} u_\nu$$

$$\frac{\partial}{\partial x^\nu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right) = \left(\frac{1}{c}\frac{\partial}{\partial \tau}, \nabla\right)$$

$\underline{\mu = 0, \nu = 0}$

$$\frac{dp^0}{d\tau} = 0$$

$$\Rightarrow \frac{dE}{dt} = 0$$

c) $$T^{\mu\nu}(x) = \int_{-\infty}^{+\infty} d\tau \, m_0 c \frac{dx^\mu(\tau)}{d\tau} \frac{dx^\nu(\tau)}{d\tau} \delta^4(x - x(\tau))$$

We want $$\partial_\nu T^{\mu\nu} = \frac{1}{c} F^{\mu\nu} j_\nu = F^{\mu\nu} \int_{-\infty}^{+\infty} d\tau \, e \frac{dx_\nu}{d\tau} \delta^4(x - x(\tau))$$

$$\partial_\nu T^{\mu\nu} = \int_{-\infty}^{+\infty} d\tau \, m_0 c \frac{\partial}{\partial x^\nu} \frac{\partial x^\mu(\tau)}{\partial \tau} \frac{\partial x^\nu(\tau)}{\partial \tau} \delta^4(x - x(\tau))$$

$$\frac{1}{c}\frac{\partial}{\partial \tau}\left(\frac{\partial x^\mu(\tau)}{\partial \tau}\right)$$

$$\frac{1}{c}\frac{\partial^2 (m_0 x^\mu)}{\partial \tau^2} = \frac{\partial p^\mu}{\partial \tau} = \frac{e}{c} F^{\mu\nu} u_\nu$$

$$= \int_{-\infty}^{+\infty} d\tau \, c \, F^{\mu\nu} \frac{dx_\nu}{d\tau} \frac{e}{c} \delta^4(x - x(\tau)) = \frac{1}{c} F^{\mu\nu} j_\nu$$