Electricity & Magnetism Qualifier

For each problem state what system of units you are using.

1. Imagine that a spherical balloon is being filled with a charged gas in such a way that the *rate* of charge being introduced is constant; $\dot{q} \equiv \frac{dq}{dt} =$ constant. Furthermore, assume that the elasticity of the balloon is such that the volume charge *density*, which you can assume to be uniform, remains constant; $\rho(t) =$ constant. For a fixed point a inside the balloon, $a \leq R$, where R is the balloon radius, calculate the following quantities as a function of the given variables, a, R, ρ, and $\dot{q}$:

    (a) [3 points] The electric field $\vec{\mathbf{E}}$.

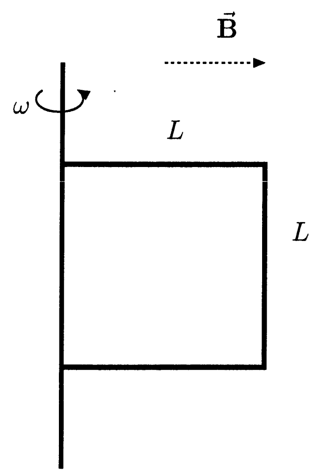
    (b) [3 points] The electric potential V.

    (c) [2 points] The time rate of change of $\vec{\mathbf{E}}$.

    (d) [2 points] The time rate of change of V.

2. A square conducting loop of wire with sides of length L, rotates (without friction) about a vertical axis through one side or edge of the square loop. The plane of the loop is perpendicular to the vertical axis. The wire in the loop has a cross-sectional area A, a mass density μ, and an electrical resistivity ρ. There is a constant horizontal magnetic field B perpendicular to the axis of rotation. The angular velocity ω of the loop decreases slowly compared with the period of rotation; let ω_0 be the angular velocity of the loop at time $t = 0$. Find the quantities requested in parts a through e below in terms of the variables: L, A, B, μ, ρ, ω, ω_0, and t. (*Note:* μ *has units of mass per unit volume*)

Helpful hint: [The moment of inertia of a horizontal rod of mass m and length L rotating about a vertical axis at one end of the rod is $(mL^2/3)$*.] OR [The moment of inertia of a square loop of wire with total mass m and a side of length L, which is rotating about an axis co-incident with one side, is* $(5mL^2/12)$*.]*



(a) [1 point] Find the kinetic energy of the rotating loop of wire.

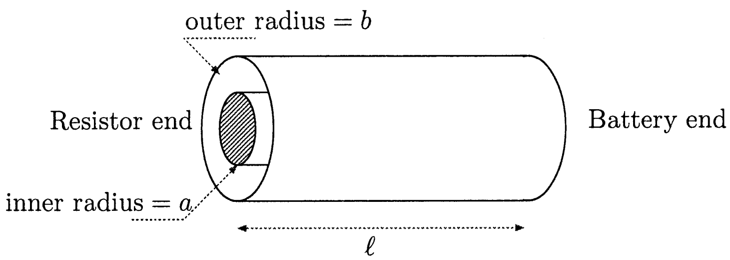
(b) [2 points] Find the current I induced in the rotating loop.

(c) [3 points] Find the average power $< P >$ dissipated by the current in the loop over one period.

(d) [3 points] Using parts a and c, derive an equation for the time dependence of ω.

(e) [1 point] Find the time T at which $\omega/\omega_0 = 1/e$.

3. A long coaxial cable of length ℓ and resistance per unit length R_ℓ, consists of an inner conductor of radius a, and an outer conductor of inner radius b. A battery of voltage V is connected between the inner and outer conductors at one end of the cable and a resistor R_0 is placed between the inner and outer conductors at the other end. The inner conductor carries a steady current I to the right. The outer conductor carries the same current I in the opposite direction. *Ignore edge effects.*



(a) [3 points] Calculate the magnetic field everywhere.

(b) [5 points] In order for the current to flow against the resistivity of the conductor, an electric field given by $\vec{\mathbf{E}} = -IR_\ell\hat{\mathbf{z}}$ must exist inside the inner conductor ($\hat{\mathbf{z}}$ pointing along the conductor from the resistor end ($z = 0$) of the cable to the battery end ($z = \ell$).). Using appropriate boundary conditions, this leads to the potential inside the inner conductor being $V(r < a, z) = I(R_0 + R_\ell z)$. In addition, the potential of the outer conductor can be taken as $V(r = b, z) = 0$. Using this information and the appropriate boundary conditions, calculate the electric field in the region $a < r < b$.

(c) [2 points] Calculate the total electromagnetic momentum stored in the fields ignoring edge effects.

4. A source located at the origin of our coordinate system emits a sinusoidal plane polarized electromagnetic wave with a linearly increasing amplitude. Assume the amplitude of the wave at time t at the source is given by $E_m = E_0(1 + \alpha t)$ with $\alpha \ll \omega$, therefore the amplitude E_m varies slightly over one period.

   (a) [2 points] Write down expressions for the electric field intensity $\vec{\mathbf{E}}$ and the magnetic field intensity $\vec{\mathbf{B}}$ at the source.

   (b) [2 points] Write down expressions for the electric field intensity $\vec{\mathbf{E}}$ and the magnetic field intensity $\vec{\mathbf{B}}$ at a distance z from the source at time t.

   Now consider a imaginary cylinder a distance z_0 from the source, with length L and radius R centered on an axis along the direction of propagation that passes through the origin.

   (c) [2 points] Calculate the time average Poynting vector ($\vec{\mathbf{S}} = \frac{1}{\mu_0}\vec{\mathbf{E}} \times \vec{\mathbf{B}}$) over one period at an arbitrary point z. Again assume that E_0 varies slightly over one period, so that averages can be calculated as usual.

   (d) [2 points] Evaluate the time average Poynting vector at the entrance and exit of the cylindrical volume.

   (e) [2 points] Integrate the average Poynting vector over the surface area of the cylindrical volume to calculate the net average outward energy flow.

5. Recall that in the Lorentz gauge, the scalar potential V and the vector potential $\vec{\mathbf{A}}$ obey the inhomogeneous wave equations

$$\nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon_0}$$
$$\nabla^2 \vec{\mathbf{A}} - \frac{1}{c^2}\frac{\partial^2 \vec{\mathbf{A}}}{\partial t^2} = -\mu_0 \vec{\mathbf{J}}$$

with solutions

$$V(\vec{\mathbf{r}}, t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{\mathbf{r}}', t_r)}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|}\, d^3r' \qquad \vec{\mathbf{A}}(\vec{\mathbf{r}}, t) = \frac{\mu_0}{4\pi}\int \frac{\vec{\mathbf{J}}(\vec{\mathbf{r}}', t_r)}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|}\, d^3r'$$

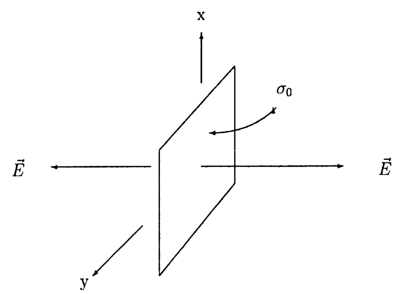
(a) [2 points] Define the retarded time t_r and briefly explain its meaning.

Suppose you take a plastic ring of radius R and glue charge on it so that the line charge density is $\lambda_0|\sin(\theta/2)|$. Then you spin the loop about its axis at an angular velocity ω.

(b) [4 points] Find the (exact) scalar potential at the center of the ring. How does your answer differ from the static ($\omega = 0$) case?

(c) [4 points] Find the (exact) vector potential at the center of the ring. How does your answer differ from the static ($\omega = 0$) case?

6. Consider a thin very large sheet of charge (as shown in the figure) such that you can ignore edge effects,



(a) [2 points] Compute the 4-current $J^\alpha(x^\beta) \equiv (c\rho, \mathbf{J})$ for the stationary sheet of charge located at $z = 0$ in the lab (see the figure). Assume the surface charge density is a constant σ_0.

(b) [2 points] Compute the E & M fields in the lab frame for the above 4-current source.

(c) [2 points] Now assume you move with speed $v < c$ in the $+x$-direction relative to the lab.
Compute the 4-current $J'^\alpha(x'^\beta)$ in your frame. You can compute J'^α from the lab's J^α by a Lorentz boost, i.e., $J'^\alpha = L^\alpha_{\ \beta} J^\beta$ or in matrix notation $J' = LJ$. If you derived J' some other way than using the above Lorentz boost, check to see that your J' satisfies $J' = LJ$.
Recall

$$L^\alpha_{\ \beta} = \left\{ \begin{array}{cccc} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right\}.$$

(d) [1 points] Combine the above E & M fields into a single E & M tensor $F^{\alpha\beta}$ defined (in Gaussian units) by

$$F^{\alpha\beta} = \left\{ \begin{array}{cccc} 0 & -E^x & -E^y & -E^z \\ E^x & 0 & -B^z & B^y \\ E^y & B^z & 0 & -B^x \\ E^z & -B^y & B^x & 0 \end{array} \right\}.$$

For SI units each $\mathbf{B}$ component in $F^{\alpha\beta}$ above is divided by c.

(e) [3 points] What will the $\mathbf{E}'$ & $\mathbf{B}'$ fields be in your frame?
You can compute them by using a Lorentz boost, i.e., $F'^{\alpha\beta} = L^\alpha_{\ \mu} L^\beta_{\ \nu} F^{\mu\nu}$ or in matrix notation $F' = LFL^T$. If you derived $\mathbf{E}'$ & $\mathbf{B}'$ some other way than boosting F show that the F' constructed with your $\mathbf{E}'$ & $\mathbf{B}'$ fields is related to the lab's $F^{\alpha\beta}$ by $F' = LFL^T$.

Prob 6

a) $\rho' = \sigma_0 \delta(z)$

$$J'^{\mu} = \begin{pmatrix} c\sigma_0 \delta(z) \\ 0 \end{pmatrix}$$

b)

Using Gauss's law

$$\int \bar{E}' \cdot d\bar{a}' = 4\pi \int \rho' \, da'$$

$$\Rightarrow 2E' \int r\,dr\,d\theta = 4\pi \int \sigma_0 \delta(z)\, r\,dr\,d\theta$$

$$\Rightarrow 2E' (\pi r'^2) = 4\pi \sigma_0 \pi r'^2$$

$$\Rightarrow \bar{E}' = 2\pi\sigma_0 \left(\Theta(z) - \Theta(-z) \right) \hat{z}$$

$$\bar{B}' = 0$$

c) $J^{\alpha} = (L^{-1})^{\alpha}{}_{\beta} J^{\beta}$

$$\begin{pmatrix} c\rho \\ \bar{J} \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} c\sigma_0 \delta(z) \\ 0 \end{pmatrix}$$

$$\rho = \gamma \sigma_0 \delta(z)$$

$$\bar{J} = \gamma\beta c \sigma_0 \delta(z) \hat{x}$$