Classical Mechanics and Statistical/Thermodynamics

January 2011

Possibly Useful Information

Handy Integrals:

$$\int_0^\infty x^n e^{-\alpha x}\,dx = \frac{n!}{\alpha^{n+1}}$$

$$\int_0^\infty e^{-\alpha x^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$$

$$\int_0^\infty x\, e^{-\alpha x^2}\,dx = \frac{1}{2\alpha}$$

$$\int_0^\infty x^2\, e^{-\alpha x^2}\,dx = \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}}$$

$$\int_{-\infty}^\infty e^{i\,a\,x - b\,x^2}\,dx = \sqrt{\frac{\pi}{b}}e^{-a^2/4b}$$

Geometric Series:

$$\sum_{n=0}^\infty x^n = \frac{1}{1-x} \qquad \text{for} \quad |x| < 1$$

Stirling's approximation:

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

Riemann and related functions:

$$\sum_{n=1}^\infty \frac{1}{n^p} \equiv \zeta(p)$$

$$\sum_{n=1}^\infty \frac{z^p}{n^p} \equiv g_p(z) \qquad\qquad \sum_{n=1}^\infty (-1)^p\,\frac{z^p}{n^p} \equiv f_p(z)$$

$$g_p(1) = \zeta(p) \qquad\qquad f_p(1) = \zeta(-p)$$

$$\zeta(1) = \infty \qquad\qquad \zeta(-1) = -\tfrac{1}{12} = 0.0833333$$

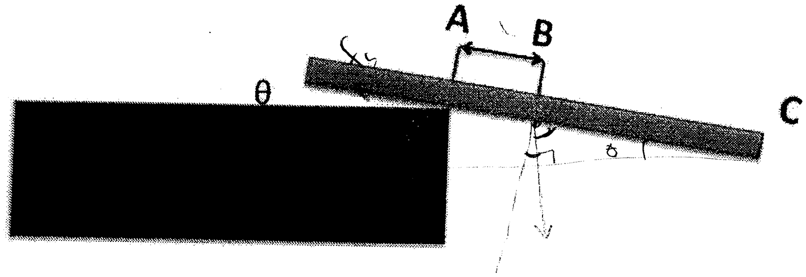
$$\zeta(2) = \tfrac{\pi^2}{6} = 1.64493 \qquad\qquad \zeta(-2) = 0$$

$$\zeta(3) = 1.20206 \qquad\qquad \zeta(-3) = \tfrac{1}{120} = 0.0083333$$

$$\zeta(4) = \tfrac{\pi^4}{90} = 1.08232 \qquad\qquad \zeta(-4) = 0$$

Classical Mechanics

1. A uniform rod of mass M and length L is held flat on a horizontal table. The center of mass, at point B, projects a distance d past the edge of the table. The rod is released at rest from a horizontal position ($\theta = 0$). The rod starts to rotate and eventually slides off the table. The forces acting on the rod are gravity, and a contact force at the edge of the table. The contact force at the table edge can be broken into two components: a normal force N that is in the plane of the page and perpendicular to the rod, and a friction force F_f that is in the plane of the page and parallel to the rod. The coefficient of static friction between the table edge and the rod is μ.



   (a) Show that the moment of inertia of a rod of length L and mass M about an axis perpendicular to the rod and passing through the center of mass is $ML^2/12$. (1pt)

   (b) Derive an expression for the moment of inertia of the rod about a general point A, a distance d from the center of mass at point B. (1pt)

   (c) When the rod is initially released, it will rotate without sliding, about the point A. Derive an expression for the angular velocity of the rotating rod, ω as a function of θ. (3pts)

   (d) Calculate the force N as a function of θ assuming that the rod does not slide, but only pivots about the point A. (3pts)

   (e) At some angle, as the rod rotates, the force of friction at the point of contact will not be able to prevent the rod from sliding. Determine the angle θ_0 at which the rod begins to slide off the table. (3pts)

1.

a) Moment of interia of the rod through the center of mass

$$\rho = \frac{m}{AX}$$

$$\Rightarrow dm = \rho A\, dx$$

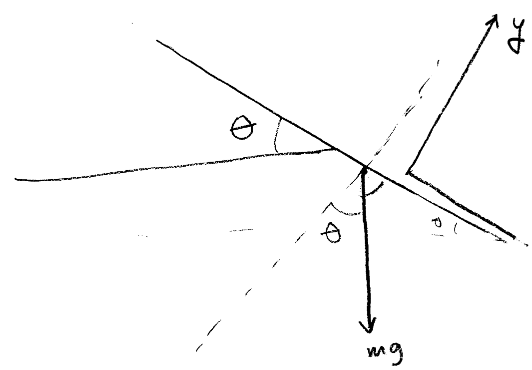
$$I = \int x^2 dm$$

$$= \rho A \int_{-L/2}^{+L/2} x^2 dx$$

$$= \rho A \left[\frac{x^3}{3}\right]_{-L/2}^{L/2}$$

$$= \rho A \left[\frac{L^3}{24} + \frac{L^3}{24}\right)$$

$$= \frac{M}{L}\ \frac{L^3}{12} = \frac{ML^2}{12}$$



b) $$I_A = I_{cm} + Md^2$$

$$= \frac{ML^2}{12} + Md^2$$

c) $$\tau_{tot} = F_N \times 0 + F_f \times 0 + \underbrace{Mg\cos\theta \times d}_{Mgd\sin 90^\circ} + \underbrace{Mg\sin\theta \times d}_{= Mgd\sin 0^\circ}$$

$$= Mgd\cos\theta = I_A \frac{d\omega}{dt}$$

$$\Rightarrow \frac{Mgd\cos\theta}{I_A} = \frac{d\omega}{d\theta}\frac{d\theta}{dt}$$

$$\Rightarrow \frac{Mgd}{I_A}\cos\theta\, d\theta = \omega\, d\omega$$

$$\Rightarrow \frac{Mgd}{I_A}\sin\theta = \frac{\omega^2}{2}$$

$$\Rightarrow \omega(\theta) = \left(\frac{2Mgd\sin\theta}{I}\right)^{1/2}$$

(d) $\sum F_y = 0$

$$\Rightarrow N - mg\cos\theta = 0$$

$\sum F_x = 0$

$$mg\sin\theta - f_s = 0$$

2. It is posited by some that there is "dark matter" that only interacts gravitationally with normal baryonic matter. Imagine a solar system immersed in a uniformly dense spherical cloud of dark matter. A planet in this solar system would experience gravitational forces from both the sun and the dark matter cloud, such that the force on the planet is given by

$$\vec{F} = \left(- \frac{k}{r^2} - b\, r \right) \hat{r}$$

where r is the radial distance of the planet to the sun, k and b are positive constants, and $\hat{r}$ is the radial **unit** vector. We will consider the limit of a "point planet," ignoring the spin and internal dynamics of the planet. You can assume that the motion of the planet is confined to a plane.

(a) Derive an expression for the Lagrangian of this system and the associated equations of motion that follow from the Euler-Lagrange equation. (2 points)

(b) Show that the angular momentum about the origin, L, is conserved. (1 point)

(c) Write an expression for the total energy of the particle E as a function of m, r, dr/dt, L, k, and b. (1 points)

(d) Assume the planet moves in a circular orbit of radius a. Derive a polynomial equation that uniquely determines a. Your answer may depend upon m, k, b, L, and E. You need not solve the equation for a. (2 points)

(e) For **nearly** circular orbits, the radial position of the planet will oscillate about the equilibrium radius, a. Determine the approximate frequency of this oscillation in the limit that the oscillations are small, and that the effect of the dark matter is small. Your answer should be in terms of m, k, b, and a. (2 points)

(f) If $b = 0$, we have standard Keplerian motion. Prove that the orbit is closed by proving that the period of small radial oscillations when $b = 0$ equals the orbital period. Discuss qualitatively what happens if $b \neq 0$. (2 points)

2.

$$\bar{F} = \left(-\frac{k}{r^2} - br\right)\hat{r} = -\frac{dV(r)}{dr}\hat{r}$$

$$\Rightarrow \quad V(r) = +\int \frac{k}{r^2}\,dr + b\int r\,dr = -\frac{k}{r} + \frac{br^2}{2}$$

a) $$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + \frac{k}{r} - \frac{br^2}{2}$$

$$\rightsquigarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$$

$$\Rightarrow \quad m\ddot{r} + \frac{k}{r^2} + br = 0$$

$$\rightsquigarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\Rightarrow \quad mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta}^2 = 0$$

b) $$p_\theta = \frac{\partial L}{\partial \dot{\theta}} \quad \text{and} \quad \frac{\partial L}{\partial \theta} = 0$$

$$\Rightarrow \dot{p}_\theta = 0$$ $\Rightarrow$ Angular momentum conserved

c) Since the Lagrangian is independent of time & the force is derived from potential (i.e. no non-conservative force), the Hamiltonian would be the total energy

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \Rightarrow \dot{r} = \frac{p_r}{m}$$

angular momentum

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \Rightarrow \dot{\theta} = \frac{p_\theta}{mr^2} = \frac{L}{mr^2}$$

$$H = \sum p_i \dot{q}_i - L$$

$$= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + \frac{k}{r} - \frac{br^2}{2}$$

$$\therefore E = \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + \frac{L^2}{2mr^2} + \frac{k}{r} - \frac{br^2}{2}$$

d) for $r = a$

$$E = \frac{L^2}{2ma^2} + \frac{k}{a} - \frac{ba^2}{2}$$

$$\Rightarrow a^2 - \frac{2k}{ab} + \left(\frac{2E}{b} - \frac{bL^2}{ma^2}\right) = 0$$

e) Define small displacement from a as

$$\eta = r - a \quad \Rightarrow \dot{\eta} = \dot{r}$$

$$\Rightarrow r = \eta + a$$

Since, θ is cyclic, I will eliminate it from the lagrangian so we get effective potent which the will be Routhian func defined as

$$R = L - p_\theta \dot{\theta}$$

$$= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - \frac{k}{r} + \frac{br^2}{2} - mr^2\dot{\theta}^2$$

$$= \frac{1}{2}m\dot{r}^2 - \underbrace{\frac{p_\theta^2}{2mr^2} + \frac{k}{r} - \frac{br^2}{2}}_{V_{eff}}$$

$$V_{eff} = \frac{p_\theta^2}{2mr^2} - \frac{k}{r} + \frac{br^2}{2}$$

$$V' = -\frac{2p_\theta^2}{2mr^3} + \frac{k}{r^2} + br$$

$$V'' = \frac{3p_\theta^2}{mr^4} - \frac{2k}{r^3} + b$$

$$\tilde{V} = \frac{3p_\theta^2}{ma^4} - \frac{2k}{a^3} + b$$

In terms of small co-ordinates

$$R = \frac{1}{2}m\dot{\eta}^2 - \frac{p_\theta^2}{2m(\eta+a)^2} - \frac{k}{\eta+a} + \frac{b}{2}(\eta+a)^2$$

$$= \frac{1}{2}m\dot{\eta}^2 - \frac{p_\theta^2}{2m(\eta+a)^2} - \frac{k}{\eta+a} + \frac{b}{2}\eta^2 + ba\eta$$

Ignoring const term in the potential

$$V_{eff} = \frac{p_\theta^2}{2m(\eta+a)^2} - \frac{k}{\eta+a} + \frac{1}{2}b\eta^2 + ba\eta$$

$$\lambda = \frac{3p_\theta^2}{m^2a^4} - \frac{2k}{ma^3}$$

$$T = \frac{1}{2}m\dot{\eta}^2$$

$$\therefore \tilde{T} = m$$

Now,

$$\frac{1}{2}\frac{\partial^2 V_{eff}}{\partial \eta^2}\Big|_{\eta=a}\eta^2 = \frac{1}{2}\frac{\partial^2}{\partial \eta^2}\left\{\frac{p_\theta^2}{2m(\eta+a)^2} - \frac{K}{\eta+a} + \frac{1}{2}b\eta^2 + ab\eta\right\}\Big|_{\eta=a}\eta^2$$

$$= \frac{1}{2}\frac{\partial}{\partial \eta}\left\{-\frac{2p_\theta^2}{2m(\eta+a)^3} - \frac{K}{(\eta+a)^2} + b\eta + ab\right\}\Big|_{\eta=a}\eta^2$$

$$= \frac{1}{2}\left\{\frac{3p_\theta^2}{m(\eta+a)^4} - \frac{2K}{(\eta+a)^3} + b\right\}\Big|_{\eta=a}\eta^2$$

$$= \frac{1}{2}\underbrace{\left\{\frac{3p_\theta^2}{16a^4} - \frac{2K}{8a^3} + b\right\}}_{\tilde{V}}\eta^2$$

but $p_\theta = ma^2\dot{\theta}$

$$= \frac{1}{2}\left\{\frac{3m^2\dot{\theta}^2}{16} - \frac{2K}{8a^3} + b\right\}\eta^2$$

So, $\det(\tilde{V} - \lambda\tilde{T}) = 0$

$$\Rightarrow \frac{3m^2\dot{\theta}^2}{8} + \frac{2K}{8a^3} - b - \lambda m = 0$$

$$\Rightarrow \lambda = \frac{3m\dot{\theta}^2}{16} - \frac{K}{4ma^3} + \frac{b}{m} = \omega^2$$

$$= \frac{36K}{16a^3} - \frac{K}{4ma^3}$$

f) $$L = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 + \frac{K}{r}$$

θ cyclic $\quad \dot{P}_\theta = 0$

$$E = \frac{1}{2} m \dot{r}^2 + \frac{P_\theta^2}{2mr^2} - \frac{K}{r}$$

At turning pts $\quad \dot{r} = 0$

$$\Rightarrow \quad E - \frac{P_\theta^2}{2mr^2} + \frac{K}{r} = 0$$

$$\Rightarrow \quad E r^2 + K r - \frac{P_\theta^2}{2m} = 0$$

$$\Rightarrow \quad r = \frac{-K \pm \sqrt{K^2 + \frac{4 P_\theta^2 E}{2m}}}{2E}$$

$$r_+ = \frac{-K + \sqrt{K^2 + \frac{2E P_\theta^2}{m}}}{E}$$

$$r_- = \frac{-K - \sqrt{K^2 + \frac{2E}{m} P_\theta^2}}{E}$$

$$a = \frac{1}{2}(r_+ + r_-) = \frac{1}{E}(-2K) = -\frac{K}{2E} = \frac{K}{2|E|}$$

$$T^2 = 4\pi^2 \frac{m}{K} a^3$$

$$= 4\pi^2 \frac{m}{K} \frac{K^3}{8E^3}$$

$$= 4\pi \frac{mK^2}{8E^3}$$

3. A particle of mass m moves in a forcefield that has the form

$$V(r,\theta,\phi) = K\frac{\cos\theta}{r^2}$$

where θ is the spherical polar coordinate and r is the radial distance.

(a) Write down the time independent Hamilton-Jacobi equation for W, Hamilton's principal function, in spherical polar coordinates. (2 points)

(b) Show that this equation can be solved by the method of separation of variables, and obtain an expression for $W = W(r,\theta,\phi;E,\alpha_1,\alpha_2)$. Your answer will involve certain integrals; you do not need to evaluate them. (2 points)

(c) Derive expressions for the momenta p_r, p_θ and p_ϕ. Use these expressions to give a physical interpretation for the separation constants α_1 and α_2. (3 points)

(d) Using the equation

$$\frac{\partial W}{\partial E} = t + \beta_1$$

find how r varies with time. Describe the radial motion for the positive and negative E, for different signs of the separation constants. You do not have to explicitly solve for $r(t)$. You should describe the values of r when $t \to \pm\infty$, or whatever suitable endpoints in time are allowed by your solutions. (3 points)

3. $$V(r,\theta,\phi) = \frac{K\cos\theta}{r^2}$$

$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{2}mr^2\sin^2\theta\,\dot{\phi}^2 - \frac{K\cos\theta}{r^2}$$

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

ϕ is cyclic $\quad \dot{p}_\phi = 0 \Rightarrow p_\phi = \alpha_\phi = \text{const.}$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\sin^2\theta\,\dot{\phi}$$

$$H = \dot{r}p_r + \dot{\theta}p_\theta + \dot{\phi}p_\phi - L$$

$$= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{2}mr^2\sin^2\theta\,\dot{\phi}^2 + \frac{K\cos\theta}{r^2}$$

$$= \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{\alpha_\phi^2}{2mr^2\sin^2\theta} + \frac{K\cos\theta}{r^2}$$

* Since the Hamiltonian is independent of time it is the total energy

$$2mE = p_r^2 + \frac{p_\theta^2}{r^2} + \frac{\alpha_\phi^2}{r^2\sin^2\theta} + \frac{2mK\cos\theta}{r^2}$$

Using Hamilton's characteristic func $W(q,\alpha)$ as a generating func

$$P_r = \frac{\partial W}{\partial r} \quad \text{and} \quad P_\theta = \frac{\partial W}{\partial \theta}$$

$$\alpha_\phi = \frac{\partial W}{\partial \phi}$$

Hence, the time independent Hamilton-Jacobi eqn

for W is

$$E = \frac{1}{2m}\left[\left(\frac{\partial W}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial W}{\partial \theta}\right)^2 + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial W}{\partial \phi}\right)^2\right] + \frac{K\cos\theta}{r^2}$$

b) Rewrite the Hamilton Jacobi Eqn as

$$\left[\left(\frac{\partial W}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial W}{\partial \theta}\right)^2 - 2mE - \frac{2mK\cos\theta}{r^2}\right] r^2\sin^2\theta = -\left(\frac{\partial W}{\partial \phi}\right)^2$$

Left hand side only depends on r & θ and the right hand side only depends on ϕ, so they has to be equal to a const

$$\left(\frac{\partial W}{\partial \phi}\right)^2 = -\alpha_2$$

Thus,

$$\left[\left(\frac{\partial W}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial W}{\partial \theta}\right)^2 - 2mE - \frac{2mK\cos\theta}{r^2}\right] r^2 \sin^2\theta = \alpha_2^2$$

$$\Rightarrow \quad r^2\left[\left(\frac{\partial W}{\partial r}\right)^2 - 2mE\right] = \frac{\alpha_2^2}{\sin^2\theta} - \left(\frac{\partial W}{\partial \theta}\right)^2 + 2mK\cos\theta$$

* Again both sides are independent of eachother and must be equal to a const.

$$+\left[\left(\frac{\partial W}{\partial \theta}\right)^2 - \frac{\alpha_2^2}{\sin^2\theta} - 2mK\cos\theta\right] = +\alpha_3$$

Hence, $W(r,\theta,\phi,E,\alpha_2,\alpha_3) = W_r(r,E) + W_\theta(\theta,\alpha_2,\alpha_3) + W_\phi(\alpha_2)$

Thus,

$$-\left[2mE - \left(\frac{\partial W}{\partial r}\right)^2\right] r^2 = -\alpha_3$$

$$\Rightarrow \quad W_r = \int \sqrt{2mE - \frac{\alpha_3}{r^2}}\, dr$$

$$\Rightarrow \quad W_\theta = \int \sqrt{\alpha_3 - \frac{\alpha_2^2}{\sin^2\theta} - 2mK\cos\theta}\, d\theta$$

$$\Rightarrow \quad W_\phi = \alpha_2 \phi$$

c) $$p_\phi = \frac{\partial W}{\partial \phi} = \alpha_2 \rightarrow$$ z-component of L and a const.

$$p_r = \frac{\partial W}{\partial r} = \sqrt{2mE - \frac{\alpha_3}{r^2}}$$

$$p_\theta = \frac{\partial W}{\partial \theta} = \sqrt{\alpha_3 - \frac{\alpha_2^2}{\sin^2\theta} - 2mK\cos\theta}$$

Now,

$$\alpha_3 = p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta} + 2mK\cos\theta = L^2 + 2mK\cos\theta$$

$\uparrow$ total angular momentum

d) $$\frac{\partial W}{\partial E} = t + \beta_1$$

$$\Rightarrow \frac{\partial W_r}{\partial E} + \frac{\partial W_\theta}{\partial E} + \frac{\partial W_\phi}{\partial E} = t + \beta_1$$

but $$\frac{\partial W_\theta}{\partial E} = 0 = \frac{\partial W_\phi}{\partial E}$$

So,

$$t + \beta_1 = \frac{\partial}{\partial E} \int \sqrt{2mE - \frac{\alpha_3}{r^2}} \, dr$$

$$= \int \frac{1}{2} \frac{2m}{\sqrt{2mE - \frac{\alpha_3}{r^2}}} \, dr$$

$$= \int \frac{m}{\sqrt{2mE - \frac{\alpha_3}{r^2}}} \, dr$$

$$= \frac{m}{\sqrt{2mE}} \int_0^\infty \frac{r \, dr}{\sqrt{r^2 - \frac{\alpha_3}{2mE}}}$$

$$r = \sqrt{\frac{\alpha_3}{2mE}} \sec\theta$$

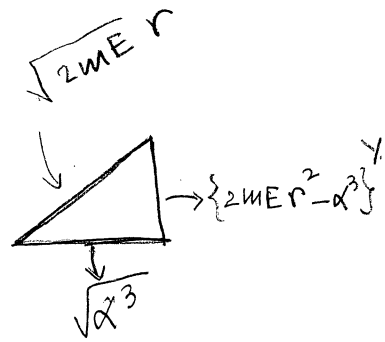
$$dr = \sqrt{\frac{\alpha_3}{2mE}} \sec\theta \tan\theta \, d\theta$$

$$= \frac{m}{\sqrt{2mE}} \int \frac{\sec^2\theta \tan\theta \, d\theta}{\tan\theta} \sqrt{\frac{\alpha_3}{2mE}}$$

$$\Rightarrow \quad t+\beta_1 = \frac{\sqrt{\alpha_3}}{2E} \int \sec^2\theta \, d\theta$$

$$\Rightarrow \quad t+\beta_1 = \frac{\sqrt{\alpha_3}}{2E} \tan\theta + \beta'$$

$$\Rightarrow \quad t+\beta = \frac{\sqrt{\alpha_3}}{2E} \tan\theta$$



Now,
$$\theta = \sec^{-1}\left\{ \sqrt{\frac{2mE}{\alpha_3}}\, r \right\}$$

$$\theta = \tan^{-1} \sqrt{\frac{2mEr^2-\alpha^3}{\alpha^3}}$$

$$\Rightarrow \quad t+\beta = \frac{1}{2E} \sqrt{2mEr^2-\alpha^3}$$

$$\Rightarrow \quad (2E)^2 (t+\beta)^2 = 2mEr^2 - \alpha^3$$

$$\Rightarrow \quad r^2 = \frac{\alpha_3}{2mE} + (t+\beta)^2 \frac{2E}{m}$$

* Nature of the motion depends on the sign of E & α_3. If $E>0$ and $\alpha_3>0$ then the particle comes from ∞. For $(t+\beta)=0$, $r=\frac{\alpha_3}{2mE}$ is minimum

S-11

Statistical Mechanics

4. The **grand free energy** or **grand potential**, Ξ, can be obtained from the Helmholtz, $F(T,V,N)$ free energy or the internal energy $U(S,V,N)$ via:

$$\Xi = F - \mu N = U - TS - \mu N$$

   (a) What are the normal or proper variables for Ξ? (When Ξ is written in terms of its normal or proper variables, it constitutes a complete thermodynamic description, without loss of information). (1 point)

   (b) Derive expressions for the conjugate variables in this description. (1 point)

   (c) What are the Maxwell relations governing derivatives of Ξ? (2 points)

   (d) Consider a small system connected to a large thermodynamic reservoir. State under what conditions (e.g. specify what quantities are exchanged between the system and reservoir) Ξ is minimized in equilibrium. Prove that this is the case by showing that Ξ is minimized when the system is in equilibrium. (3 points)

   (e) Given the Helmholtz free energy for an ideal gas:

$$F(T,V,N) = -NkT\left(1 + \log\left(\frac{VT^{3/2}}{N\Phi}\right)\right)$$

   where Φ is an unspecified but fixed constant, calculate the grand free energy for an ideal gas. (3 points)

4.

a) $\mathcal{G} = E - TS - \mu N$

$d\mathcal{G} = -PdV - SdT - Nd\mu$

So, V, T & μ are the proper variables

b) $P = -\left.\frac{\partial \mathcal{G}}{\partial V}\right|_{T,\mu}$

$S = -\left.\frac{\partial \mathcal{G}}{\partial T}\right|_{V,\mu}$

$N = -\left.\frac{\partial \mathcal{G}}{\partial \mu}\right|_{V,T}$

c) $\rightsquigarrow \quad -\frac{\partial}{\partial T}\left(\frac{\partial \mathcal{G}}{\partial V}\right) = -\frac{\partial}{\partial V}\left(\frac{\partial \mathcal{G}}{\partial T}\right)$

$\Rightarrow \quad \boxed{\frac{\partial P}{\partial T} = \frac{\partial S}{\partial V}}$

$\rightsquigarrow \quad -\frac{\partial}{\partial \mu}\left(\frac{\partial \mathcal{G}}{\partial V}\right) = -\frac{\partial}{\partial V}\left(\frac{\partial \mathcal{G}}{\partial \mu}\right)$

$\Rightarrow \quad \boxed{\frac{\partial P}{\partial \mu} = \frac{\partial N}{\partial V}}$

$$-\frac{\partial}{\partial T}\left(\frac{\partial \mathcal{G}}{\partial \mu}\right) = -\frac{\partial}{\partial \mu}\left(\frac{\partial \mathcal{G}}{\partial T}\right)$$

$$\boxed{\frac{\partial N}{\partial T} = \frac{\partial S}{\partial \mu}}$$

d) $d\mathcal{G} = -p\,dV - S\,dT - N\,d\mu$

At equilibrium, T & μ is fixed

So, $dT = 0 = d\mu$

$\therefore\ d\mathcal{G} = -p\,dV$

So, inorder to minimize the free

e)

$$F(T,V,N) = -NKT\left(1+\log\left(\frac{VT^{3/2}}{N\Phi}\right)\right)$$

$$\mathcal{G} = F - \mu N$$

$$\mu = \left.\frac{\partial F}{\partial N}\right|_{T,V} = \frac{\partial}{\partial N}\left\{-NKT + NKT\log\left(\frac{N\Phi}{VT^{3/2}}\right)\right\}$$

$$= \frac{\partial}{\partial N}\left\{-NKT + NKT\log N + NKT\log\left(\frac{\Phi}{VT^{3/2}}\right)\right.$$

$$= -\cancel{KT} + KT\log N + \cancel{KT} + KT\log\left(\frac{\Phi}{VT^{3/2}}\right)$$

$$\frac{\mu}{KT} = \log\left(\frac{N\Phi}{VT^{3/2}}\right)$$

$$\Rightarrow N = \frac{VT^{3/2}}{\Phi}e^{\mu/KT}$$

$$\mathcal{G} = F - \frac{\mu VT^{3/2}}{\Phi}e^{\mu/KT}$$

$$= -NKT\left(1+\log\left(\frac{VT^{3/2}}{N\Phi}\right)\right) - \frac{VT^{3/2}}{\Phi}\mu e^{\mu/KT}$$

5. Consider a solid consisting of a lattice of N atoms. If an atom is knocked slightly out of its proper position in the lattice, this is called a *point defect.* Assume that the energy of a point defect is ϵ, and that if there is no defect, the energy at that site is zero. Assume further that the defects are distinguishable, and do not interact with each other.

   (a) Determine the number of ways to place n point defects within the N lattice sites. From this show that the entropy associated with their configuration is approximately

   $$S(N,n) \sim k\left[N\ln N - n\ln n - (N-n)\ln(N-n)\right]$$

   (3 points).

   (b) Derive a (simple) expression for the internal energy, U, of the system as a function of n. From this result, write the entropy above, $S(N,n)$ as function of U and N, obtaining $S(U,N)$. (2 points).

   (c) Using the expression for $S(U,N)$ above, calculate the chemical potential for changes in N. (3 points).

   (d) In thermal equilibrium we know

   $$T = \left[\frac{\partial S}{\partial U}\right]^{-1}$$

   Use the approximate results above to calculate this temperature as a function of U and N. From this result to find the number of defects, $n(N,T)$. (2 points).

5.

a) $$\Omega = \binom{N}{n} = \frac{N!}{n!\,(N-n)!}$$

$$\ln N! = N\ln N - N$$

$$S = K\ln\Omega$$

$$= K\{\ln N! - \ln n! - \ln (N-n)!\}$$

$$= K\{N\ln N - \cancel{N} - n\ln n + \cancel{n} - (N-n)\ln(N-n) + \cancel{N} - \cancel{n}\}$$

$$= K[N\ln N - n\ln n - (N-n)\ln(N-n)]$$

b) $$U = n\epsilon \Rightarrow n = \frac{U}{\epsilon}$$

$$S(U,N) = K\left[N\ln N - \frac{U}{\epsilon}\ln\left(\frac{U}{\epsilon}\right) - \left(N-\frac{U}{\epsilon}\right)\ln\left(N-\frac{U}{\epsilon}\right)\right]$$

c) $$F = U - TS$$

$$F = U - KT\left[N\ln N - \frac{U}{\epsilon}\ln\left(\frac{U}{\epsilon}\right) - \left(N-\frac{U}{\epsilon}\right)\ln\left(N-\frac{U}{\epsilon}\right)\right.$$

$$F = U - TS$$

$$\Rightarrow \quad dF = -PdV + \mu dN - SdT$$

$$\Rightarrow \quad \mu = \frac{\partial F}{\partial N}\bigg|_{V,T}$$

$$\mu = -KT \frac{\partial}{\partial N}\left\{ N\ln N - N\ln\left(N - \frac{U}{\epsilon}\right) + \frac{U}{\epsilon}\ln\left(N - \frac{U}{\epsilon}\right) \right\}$$

$$= -KT\left\{ \ln N + 1 - \ln\left(N - \frac{U}{\epsilon}\right) - \frac{N}{N - \frac{U}{\epsilon}} + \frac{U/\epsilon}{N - U/\epsilon} \right\}$$

$$= -KT\left\{ \ln\left(\frac{N}{N - U/\epsilon}\right) + \cancel{1} - \frac{\cancel{(N - U/\epsilon)}}{\cancel{(N - U/\epsilon)}} \right\}$$

$$= -KT \ln\left(\frac{N}{N - U/\epsilon}\right)$$

d)

$$T = \left[\frac{\partial S}{\partial U}\right]^{-1}$$

$$= \left\{ \frac{\partial}{\partial U} K\left[N\ln N - \frac{U}{\epsilon}\ln\left(\frac{U}{\epsilon}\right) - \left(N - \frac{U}{\epsilon}\right)\ln\left(N - \frac{U}{\epsilon}\right) \right\}^{-1}$$

$$= \frac{1}{K}\left[-\frac{\cancel{U}}{\epsilon}\left(\frac{\epsilon}{\cancel{U}}\right)\left(\frac{1}{\epsilon}\right) - \frac{N}{N - \frac{U}{\epsilon}}\left(-\frac{1}{\epsilon}\right) + \frac{1}{\epsilon}\ln\left(N - \frac{U}{\epsilon}\right) - \frac{1}{\epsilon}\ln\left(\frac{U}{\epsilon}\right) + \frac{U/\epsilon}{N - U/\epsilon}\left(-\frac{1}{\epsilon}\right) \right]^{-1}$$

$$T = \frac{1}{K}\left[-\frac{1}{\epsilon} - \frac{1}{\epsilon}\ln\left(\frac{U}{\epsilon}\right) + \frac{N}{\epsilon N - U} + \frac{1}{\epsilon}\ln\left(N - \frac{U}{\epsilon}\right) - \frac{U}{\epsilon^2 N - U\epsilon}\right]^{-1}$$

$$\Rightarrow -\frac{1}{\epsilon} + \frac{1}{\epsilon}\ln\left(\frac{N - U/\epsilon}{U/\epsilon}\right) + \frac{N}{\epsilon N - U} - \frac{U}{\epsilon N - U}\left(\frac{1}{\epsilon}\right) = \frac{1}{KT}$$

$$\Rightarrow -\frac{1}{\epsilon} + \frac{1}{\epsilon}\ln\left(\frac{N-n}{n}\right) + \frac{1}{\epsilon}\frac{N}{N-n} - \frac{1}{\epsilon}\frac{n\cancel{\epsilon}}{\cancel{\epsilon}(N-n)} = \frac{1}{KT}$$

$$\Rightarrow \frac{1}{\epsilon}\left(\frac{N}{N-n} - 1 - \frac{n}{N-n}\right) - \frac{1}{\epsilon}\ln\left(\frac{N-n}{n}\right) = \frac{1}{KT}$$

$$\Rightarrow \frac{1}{\epsilon}\left(\frac{N - N + n - n}{N-n}\right) + \frac{1}{\epsilon}\ln\left(\frac{N}{n} - 1\right) = \frac{1}{KT}$$

$$\Rightarrow \ln\left(\frac{N}{n} - 1\right) = +\frac{\epsilon}{KT}$$

$$\Rightarrow \frac{N}{n} - 1 = e^{+\epsilon/KT}$$

$$\Rightarrow \frac{N}{n} = 1 + e^{+\epsilon/KT}$$

$$\Rightarrow n = \frac{N}{1 + e^{+\beta\epsilon}}$$

6. A black body may be thought of as a system of harmonic oscillators possessing all possible frequencies—equivalently, it is a system of photons governed by the Bose-Einstein distribution.

   (a) Calculate the average energy $u(\nu)$ of a quantum harmonic oscillator of frequency ν at temperature T where the allowed energies of the oscillator are:
$$E(n) = h\nu n$$
   and we have ignored the zero-point energy. (3 points)

   (b) The number of oscillators per unit phase space is $2\,d^3q\,d^3p/h^3$, where the factor of 2 comes from the two transverse polarization states of the photon. Calculate the total energy of the black body
$$U = 2\int \frac{d^3q\,d^3p}{h^3} u(\nu)$$
   in terms of a single dimensionless integral. This is the famous Planck formula. [Use the relativistic relation between frequency and momentum for photons, $h\nu = pc$.] (3 points)

   (c) Derive the Stefan-Boltzmann law, $u = aT^4$, and compute the constant a using the formula
$$\int_0^\infty dx \frac{x^{n-1}}{e^x - 1} = \zeta(n)\Gamma(n),$$
   where $\zeta(n)$ is the Riemann zeta function, and $\Gamma(n)$ is the gamma function . Your answer will be in terms of mathematical and physical constants. (4 points)

6.

a) The energy levels of one oscillator is

$$E(n) = h\nu n$$

The one oscillator partition func can be written as

$$Z_1(V,T) = \sum_{n=0}^{\infty} e^{-\beta h\nu n} = \frac{1}{1-e^{-\beta h\nu}}$$

$$\langle E \rangle = -\frac{\partial \ln Z_1}{\partial \beta} = \frac{\partial}{\partial \beta}\ln\left(1-e^{-\beta h\nu}\right) = \frac{h\nu e^{-\beta h\nu}}{1-e^{-\beta h\nu}}$$

$$u(\nu) = \frac{h\nu}{e^{\beta h\nu}-1}$$

b) For photons, $h\nu = pc$

Thus, $$U = 2\int \frac{d^3q\, d^3p}{h^3} \frac{pc}{e^{\beta pc}-1}$$

$$= \frac{2V}{h^3} 4\pi \int_0^{\infty} dp \frac{p^3 c}{e^{\beta pc}-1}$$

$$= \frac{8\pi V}{h^3} \int_0^{\infty} dp \frac{p^3 c}{e^{\beta pc}-1}$$

$\beta pc = x$

$dp = \frac{1}{\beta c} dx$

$p^3 = \frac{1}{\beta^3 c^3} x^3$

$$= \frac{8\pi V \beta^4}{h^3 c^3} \int_0^{\infty} \frac{x^3}{e^x - 1} dx$$

c)

$$\int_0^\infty dx \; \frac{x^{n-1}}{e^x - 1} = \zeta(n)\,\Gamma(n)$$

$$\rightsquigarrow \int_0^\infty dx \; \frac{x^3}{e^x - 1} = \zeta(4)\,\Gamma(4) = \frac{\pi^4}{90}\,3! = \frac{\pi^4}{15}$$

thus,

$$\frac{U}{V} = \frac{8\pi k^4 T^4}{h^3 c^3}\,\frac{\pi^4}{15}$$

$$\Rightarrow \quad u = a\,T^4$$