Classical Mechanics and Statistical/Thermodynamics

January 2008

Possibly Useful Information

Handy Integrals:

$$\int_0^\infty x^n e^{-\alpha x}\,dx = \frac{n!}{\alpha^{n+1}}$$

$$\int_0^\infty e^{-\alpha x^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$$

$$\int_0^\infty x\;e^{-\alpha x^2}\,dx = \frac{1}{2\alpha}$$

$$\int_0^\infty x^2\;e^{-\alpha x^2}\,dx = \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}}$$

$$\int_{-\infty}^\infty e^{i\,a\,x-b\,x^2}\,dx = \sqrt{\frac{\pi}{b}}e^{-a^2/4b}$$

Geometric Series:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad \text{for} \quad |x|<1$$

Stirling's approximation:

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

Riemann and related functions:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \equiv \zeta(p)$$

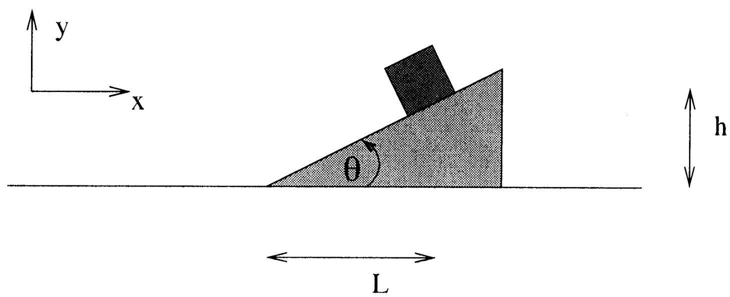
$$\sum_{n=1}^{\infty} \frac{z^n}{n^p} \equiv g_p(z) \qquad \sum_{n=1}^{\infty}(-1)^p\;\frac{z^n}{n^p} \equiv f_p(z)$$

$$g_p(1) = \zeta(p) \qquad f_p(1) = \zeta(-p)$$

$$\zeta(1) = \infty \qquad \zeta(-1) = 0.0833333$$
$$\zeta(2) = 1.64493 \qquad \zeta(-2) = 0$$
$$\zeta(3) = 1.20206 \qquad \zeta(-3) = 0.0083333$$
$$\zeta(4) = 1.08232 \qquad \zeta(-4) = 0$$

Classical Mechanics

1. A block of mass m_1 sits atop a triangular wedge of mass m_2, which is itself on a frictionless plane, as shown. The two are initially at rest, and the block is a height h above the surface of the plane, a horizontal distance L from the bottom edge of the wedge. The wedge has an opening angle θ, as shown.



   (a) Assume that there is no friction between the block and the wedge. The block slides down the wedge. What are the velocities (measured with respect to the fixed inertial reference frame denoted by the x and y axes shown) of the block and wedge just as the block reaches the lower edge of the wedge? (3 points).

   (b) Now replace the block by a ball of radius R (and mass m_1). The ball rolls down the wedge without slipping. What are the velocities of the ball and wedge just as the ball reaches the lower edge of the wedge? (3 points).

   (c) Return to the block problem, but now assume that the coefficients of static and kinetic friction between the block and the wedge are μ (they have the same value). What is μ_{min}, the minimum value of μ for which the system is stable? (1 point).

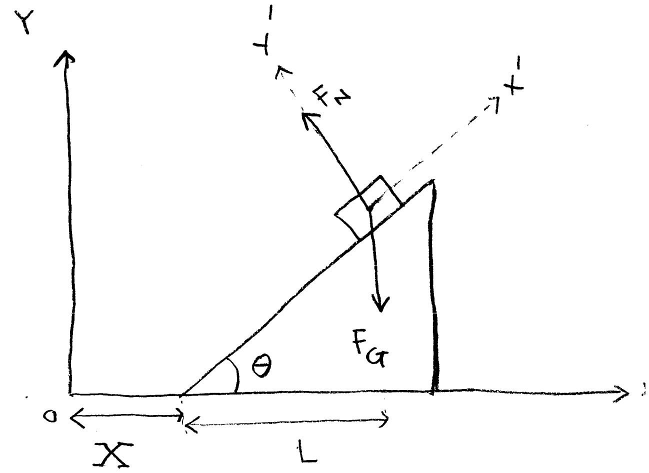
   (d) If $\mu < \mu_{\mathrm{min}}$, calculate the minimum **horizontal** force that can be applied to the wedge such that the block will not accelerate down the wedge. (3 points).

Note: you can neglect the finite size of the block in your calculation, and you are asked for the velocities before the block or ball make contact with the frictionless plane.

1.

a)

* Since there is no friction between the wedge & the horizontal surface, when the block is placed on the ramp, the horizontal component of the normal force from the block will push the wedge backward. Since, the wedge moves it will contribute to both x&y components of the acceleration of the block.



→ In X-Y ref frame,

$$a_X^{wedge} = \ddot{X} \qquad a_Y^{wedge} = 0$$

→ In x'-y' ref frame (attached to the wedge & coincides with centre of mass of the block at height h)

$$a_{x'} = +\ddot{X}\cos\theta$$

$$a_{y'} = \ddot{X}\sin\theta$$

Thus, the acceleration of the block due to the acceleration of the wedge would be equal & opposite

Using Newton's 2nd law on the block in x'-y' frame

$$\sum_x F_x = m_1 \ddot{x}'$$

$$\Rightarrow \quad -m_1 g\sin\theta = m_1(\ddot{x}' + \ddot{X}\cos\theta)$$

$$\ddot{x}' = -g\sin\theta - \ddot{X}\cos\theta$$

$$\Sigma F_y = m_1\ddot{y}' \qquad \text{as } \ddot{y}' = -\ddot{X}\sin\theta$$

$$\Rightarrow F_N' - m_1 g\cos\theta = m_1(-\ddot{X}\sin\theta)$$

$$\Rightarrow F_N' = m_1 g\cos\theta - m_1\ddot{X}\sin\theta$$

Now, $F_{N,x} = F_N'\sin\theta$

For the wedge using Newton's 2nd law in X-Y frame

$$\Sigma F_x = m_2\ddot{X}$$

$$\Rightarrow m_1(g\cos\theta - \ddot{X}\sin\theta)\sin\theta = m_2\ddot{X}$$

$$\Rightarrow \ddot{X} = \frac{m_1 g\sin\theta\cos\theta}{m_2 + m_1\sin^2\theta}$$

Thus,

$$\ddot{x}' = -g\sin\theta - \frac{m_1 g\sin\theta\cos^2\theta}{m_2 + m_1\sin^2\theta}$$

$$= \frac{-m_2 g\sin\theta - m_1 g\sin^3\theta - m_1 g\sin\theta\cos^2\theta}{m_2 + m_1\sin^2\theta}$$

$$= -\frac{(m_1+m_2)g\sin\theta}{m_2 + m_1\sin^2\theta}$$

$$a_x^{block} = \ddot{X} + \ddot{x}\cos\theta$$

$$= \frac{m_1 g \sin\theta\cos\theta}{m_2 + m_1\sin^2\theta} - \frac{(m_1+m_2)\, g\sin\theta\cos\theta}{m_2 + m_1\sin^2\theta}$$

$$= \frac{-m_2 g\sin\theta\cos\theta}{m_2 + m_1\sin^2\theta}$$

$$a_Y^{block} = \ddot{x}\sin\theta$$

$$= -\frac{(m_1+m_2)\, g\sin^2\theta}{m_2 + m_1\sin^2\theta}$$

$$a^{block} = \left[\frac{(m_2 g\cos\theta)^2\sin^2\theta}{(m_2+m_1\sin^2\theta)^2} + \frac{(m_1+m_2)^2 g^2\sin^4\theta}{(m_2+m_1\sin^2\theta)^2}\right]^{1/2}$$

$$= \left[\frac{m_1^2 g^2\sin^4\theta + m_2^2 g^2\sin^2\theta + 2m_1m_2 g^2\sin^4\theta}{(m_2+m_1\sin^2\theta)^2}\right]^{1/2}$$

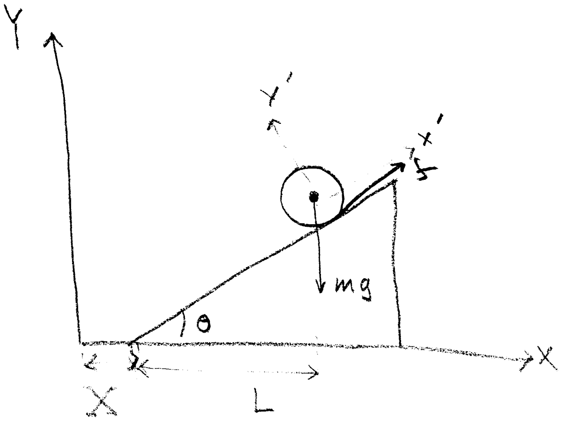
$$= \frac{\sqrt{m_2^2 + m_1\sin^2\theta\,(m_1+2m_2)}\; g\sin\theta}{m_2+m_1\sin^2\theta}$$

$$v_f^2 = v_i^2 + 2aL$$

$$\Rightarrow v_f^2 = \frac{\sqrt{m_2^2 + m_1^2 \sin^2\theta (m_1 + 2m_2)}}{m_2 + m_1 \sin^2\theta} \, g \sin\theta \, L$$

b) $a_{x'} = \ddot{X} \cos\theta$

$a_{y'} = \ddot{X} \sin\theta$



$$f_s - m_1 g \sin\theta = m_1 \ddot{x}' + m_1 \ddot{X} \cos\theta$$

$$F_N' - m_1 g \cos\theta = - m_1 \ddot{X} \sin\theta$$

$$\tau_{net} = 0 \times F_G + 0 \times F_N + r \times f_s \qquad \alpha = \ddot{x}' R$$

$$\tau_{net} = R f_s = I\alpha = \ddot{x}' R \underbrace{I}_{= \frac{2}{5} m_1 R^2}$$

$$\Rightarrow \underline{f_s = \frac{2}{5} m_1 R^2 \ddot{x}'}$$

$$\frac{2}{5} m_1 R^2 \ddot{x}' - m_1 g \sin\theta = m_1 \ddot{x}' + m_1 \ddot{X} \cos\theta$$

$$\Rightarrow m_1 \ddot{x}' \left(\frac{2R^2}{5} - 1\right) = m_1 (g \sin\theta + \ddot{X} \cos\theta)$$

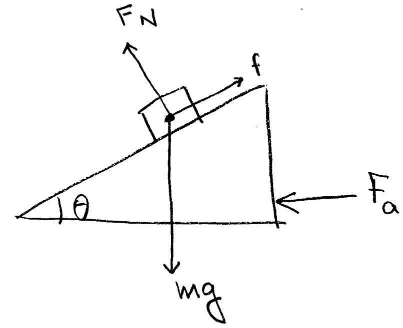
$$\Rightarrow \ddot{x}' = \frac{g \sin\theta + \ddot{X} \cos\theta}{\left(\frac{2}{5} R^2 - 1\right)}$$

$$F_N' = m (g \cos\theta - \ddot{X} \sin\theta)$$

$$\Rightarrow m (g \cos\theta - \ddot{X} \sin\theta) \sin\theta = M \ddot{X}$$

$$\ddot{X} = \frac{mg\cos\theta\sin\theta}{M + m\sin^2\theta}$$

c) $\Sigma F_x = 0$



$f_s - mg\sin\theta = 0$

$\Sigma F_y = 0$

$\Rightarrow N - mg\cos\theta = 0$

Thus, $\mu_s mg\cos\theta - mg\sin\theta = 0$

$\Rightarrow (\mu_s)_{min} = \tan\theta$

<u>For wedge</u>

d) $F_N \sin\theta - F_a - f_s = 0$ this cant be true, the wedge has to accelerate in the opposite direction

$F_a = F_N \sin\theta - f_s$

$= mg\cos\theta\sin\theta - mg\sin\theta$

$= mg\sin\theta(\cos\theta - 1)$

<u>For block</u> (In x'-y')

$\Sigma F_x = 0$

$\Rightarrow -mg\sin\theta + f_s = 0$

$\Rightarrow f_s = mg\sin\theta$

$\Sigma F_y = 0$

$F_N - mg\cos\theta = 0$

$\Rightarrow F_N = mg\cos\theta$

d) <u>For wedge</u>

* the acceleration of the wedge has to compensate for the acceleration of the block, so the net acceleration of the block is zero

$$F_N \sin\theta - F_a - f_s = m_2 \ddot{X} \Rightarrow F_a = F_N \sin\theta - f_s - m_2 \ddot{X}$$

<u>For block</u> $(\text{In } x'\text{-}y')$

$(m_1 \ddot{x} = -m_1 \ddot{X} \cos\theta)$

$$-mg\sin\theta + f_s = m_1 \ddot{X}\cos\theta \Rightarrow f_s = m_1 g \sin\theta + m_1 \ddot{X}\cos\theta$$

$$F_N - mg\cos\theta = -m_1 \ddot{X}\sin\theta$$

$$\Rightarrow F_N = m_1 g \cos\theta - m_1 \ddot{X}\sin\theta$$

but $f_s = \mu N$

$$\Rightarrow m_1 g \sin\theta + m_1 \ddot{X}\cos\theta = \mu m_1 g\cos\theta - \mu m_1 \ddot{X}\sin\theta$$

$$\Rightarrow m_1 \ddot{X}(\cos\theta + \mu\sin\theta) = m_1 g(\mu\cos\theta - \sin\theta)$$

$$\Rightarrow \ddot{X} = \frac{g(\mu\cos\theta - \sin\theta)}{(\cos\theta + \mu\sin\theta)}$$

$$= -\frac{g(\sin\theta - \mu\cos\theta)}{(\mu\sin\theta + \cos\theta)}$$

So, $F_a = (\sin\theta - \mu) F_N - m_2 \ddot{X}$

$$F_a = (\sin\theta - \mu)\left[m_1 g\cos\theta + \frac{m_1 g(\sin\theta - \mu\cos\theta)\sin\theta}{(\mu\sin\theta + \cos\theta)}\right] + \frac{m_2 g(\sin\theta - \mu\cos\theta)}{(\mu\sin\theta + \cos\theta)}$$

2. Consider a point particle of mass m constrained to move on a parabola in the x-z plane, i.e.,

$$z = \frac{\alpha}{2} x^2.$$

Assume the constraint force is frictionless and gravity acts vertically ($F_z = -mg$).

(a) Use Lagrangian mechanics to write a second order differential equation for $x(t)$. (2 points)

(b) Find a first integral of this equation (any way you can) and evaluate the constant of integration using the maximum value x_{max} reached by x. (4 points)

(c) Assume that the particle is pulled a short distance from the origin and allowed to oscillate. Calculate the period in the limit of small oscillations, $\epsilon \equiv \alpha x_{\max} << 1$. (4 points)

2.

a) $L = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} m\dot{z}^2 - mgz$

$$z = \frac{\alpha}{2}x^2 \Rightarrow \dot{z} = \alpha x \dot{x}$$

$$L = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m\alpha^2 x^2 \dot{x}^2 - \frac{1}{2} mg\alpha x^2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\Rightarrow \frac{d}{dt}\left(m\dot{x} + m\alpha^2 x^2 \dot{x}\right) - m\alpha^2 x \dot{x}^2 + mg\alpha x = 0$$

$$\Rightarrow m\ddot{x} + 2m\alpha^2 x \dot{x}^2 + m\alpha^2 x^2 \ddot{x} - m\alpha^2 x \dot{x}^2 + mg\alpha x = 0$$

$$\Rightarrow \ddot{x} + \alpha^2 x^2 \ddot{x} + \alpha^2 x \dot{x}^2 + g\alpha x = 0$$

b) $$\ddot{x} = \frac{d\dot{x}}{dt} = \frac{d\dot{x}}{dx}\frac{dx}{dt} = \dot{x}\frac{d\dot{x}}{dx}$$

$$\dot{x}\frac{d\dot{x}}{dx} + \alpha^2 x^2 \dot{x}\frac{d\dot{x}}{dx} + \alpha^2 x \dot{x}^2 + g\alpha x = 0$$

$$\Rightarrow \dot{x}\frac{d\dot{x}}{dx}\left(1 + \alpha^2 x^2\right) = -\left(\alpha^2 \dot{x}^2 + g\alpha\right) x$$

$$\Rightarrow \frac{\dot{x}}{\alpha^2 \dot{x}^2 + g\alpha} d\dot{x} = -\frac{x}{1+\alpha^2 x^2} dx$$

$$\Rightarrow \int \frac{\dot{x}}{\alpha^2 \dot{x}^2 + g\alpha} \, d\dot{x} = - \int \frac{x}{1+\alpha^2 x^2} \, dx$$

$$f(x) = 1 + \alpha^2 x^2$$

$$f(\dot{x}) = \alpha^2 \dot{x}^2 + g\alpha$$

$$f'(x) = 2\alpha^2 x$$

$$f'(\dot{x}) = 2\alpha^2 \dot{x}$$

$$\int \frac{f'(x)}{f(x)} \, dx = \ln f(x) + \text{Const}$$

$$\Rightarrow \frac{1}{2\alpha^2} \int \frac{2\alpha^2 \dot{x}}{\alpha^2 \dot{x}^2 + g\alpha} \, d\dot{x} = - \frac{1}{2\alpha^2} \int \frac{2\alpha^2 x}{1+\alpha^2 x^2} \, dx$$

$$\Rightarrow \ln(\alpha^2 \dot{x}^2 + g\alpha) + \text{const} = - \ln(1+\alpha^2 x^2) + \text{const}$$

$$\Rightarrow (\alpha^2 \dot{x}^2 + g\alpha)(1+\alpha^2 x^2) = e^A \Longrightarrow$$

$$\rightsquigarrow \text{At} \quad x = x_{max} \quad \dot{x} = 0$$

$$\Rightarrow g\alpha(1+\alpha^2 x_{max}^2) = e^A$$

$$\Rightarrow A = \ln\left[g\alpha(1+\alpha^2 x_{max}^2)\right]$$

$$X = \frac{1}{\alpha}\left(\frac{e^{A}}{\alpha^2 \dot{x}^2 + g\alpha} - 1\right)^{1/2}$$

$$\uparrow$$

$$\text{or} \quad \dot{x}(t) = \frac{1}{\alpha}\left(\frac{g\alpha(1+\alpha^2 x_{max}^2)}{1+\alpha^2 x^2} - g\alpha\right)^{1/2}$$

$$\eta = x - x_0 = x \Rightarrow \dot{\eta} = \dot{x}$$

c) $$L = \frac{1}{2} m \dot{\eta}^2 + \frac{1}{2} m \alpha^2 \eta^2 \dot{\eta}^2 - \frac{1}{2} m g \alpha \eta^2$$

$$= \frac{1}{2} m \dot{\eta}^2 (1 + \alpha^2 \eta^2) - \frac{1}{2} m g \alpha \eta^2$$

$$\tilde{T} = m(1 + \alpha^4 x_{max}^2)$$

$$V_{eff} = \frac{1}{2} m g \alpha \eta^2$$

$$\frac{1}{2} \frac{\partial^2 V_{eff}}{\partial \eta^2}\bigg|_{\eta=0} \eta^2 = \frac{1}{2} \frac{\partial^2}{\partial \eta^2}\left\{\frac{1}{2} m g \alpha \eta^2\right\}\bigg|_{\eta=0} \eta^2 = \frac{1}{2}(mg\alpha)\eta^2$$

$$\tilde{V} = mg\alpha$$

$$\det, (\tilde{V} - \lambda \tilde{T}) = 0$$

$$mg\alpha - \lambda m(1+\alpha^3 x_{max}) = 0$$

$$\Rightarrow \lambda = \frac{g\alpha}{1+\alpha^4 x_{max}^2} = \omega$$

c) $L = \frac{1}{2} m \dot{\epsilon}^2 + \frac{1}{2} m \alpha^2 \epsilon^2 \dot{\epsilon}^2 - \frac{1}{2} m g \alpha \epsilon^2$

$$= \frac{1}{2} m \underbrace{(1 + \alpha^4 x_{max}^2)}_{\tilde{T}} - \frac{1}{2} \underbrace{mg\alpha}_{\tilde{V}} \epsilon^2$$

$$mg\alpha - \lambda m (1 + \alpha^4 x_{max}^2) = 0$$

$$\Rightarrow \quad \lambda = \frac{g\alpha}{(1 + \alpha^4 x_{max}^2)}$$

3. **Angular momentum and the Rungé-Lenz vector:** Given a point particle of mass m, trajectory $\vec{r}(t)$, and momentum $\vec{p}(t)$, we can define the angular momentum

$$\vec{L} \equiv \vec{r} \times \vec{p}$$

and the Rungé-Lenz vector

$$\vec{\mathcal{A}} = \frac{1}{m}\vec{p} \times \vec{L} - \hat{r}$$

We consider the explicit case of a $1/r$ potential, so that

$$H = \frac{p^2}{2m} - \frac{1}{r}$$

(a) Prove that the Poisson bracket of H and $\vec{L}$ is zero, that is:

$$\left\{H, \vec{L}\right\} = 0.$$

(3 points).

(b) Prove that the Poisson bracket of H and $\vec{\mathcal{A}}$ is zero, that is:

$$\left\{H, \vec{\mathcal{A}}\right\} = 0.$$

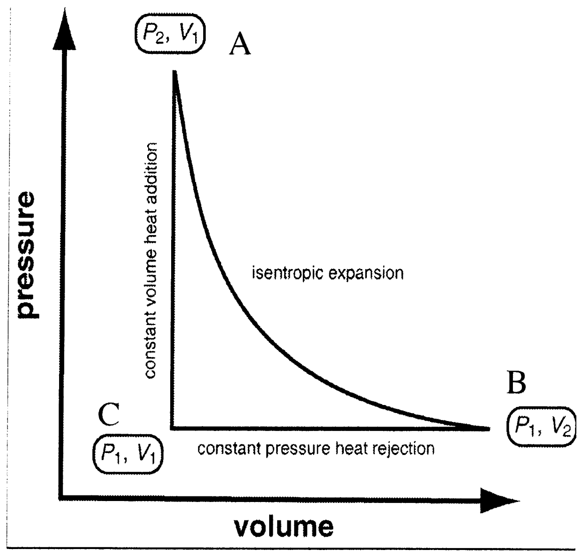
(3 points)

(c) What do your results in parts (a) and (b) imply about the behavior of $\vec{\mathcal{A}}$ and $\vec{L}$? (1 point)

(d) Evaluate $\vec{r} \cdot \vec{\mathcal{A}} = r\mathcal{A}\cos\theta$, using the explicit form for $\vec{\mathcal{A}}$ above. Use this to calculate the orbital motion of the particle (that is, a relationship between r and θ as the particle moves about its orbit). (3 points)

Statistical Mechanics

4. **Heat Engines:** A pulse jet operates under a Lenoir cycle. This consists of an adiabat, an isobar, and an isochore, as shown.



Assuming that the working fluid is an ideal 3D monoatomic gas of N particles:

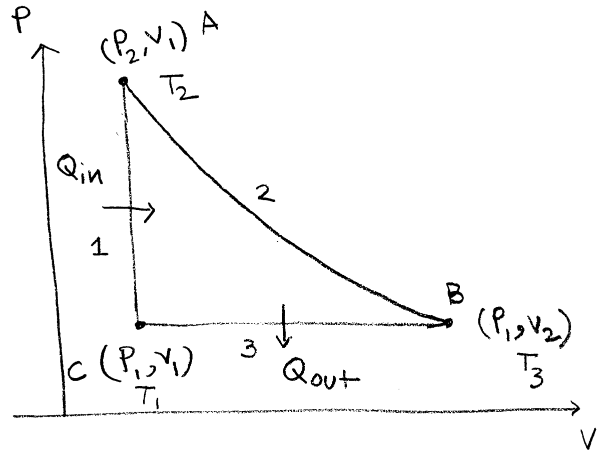
(a) Find the work done in one complete cycle. (3 points)

(b) Find the heat exchanged in each step in the cycle. (3 points)

(c) Find the efficiency of the engine. Express your answer in terms of pressures and volutmes. (3 points)

(d) To produce work, should the engine cycle operate clockwise ($A \to B \to C \to A$) or counterclockwise ($A \to C \to B \to A$) ? (1 point)

4.



a) • 1 → Ischoric $W = 0$

$$Q = C_v (T_2 - T_1)$$

$$PV = NKT$$

$$P_2 V_1 = NKT_2 \quad \& \quad P_1 V_1 = NKT_1$$

$$\Rightarrow T_2 = \frac{P_2 V_1}{NK} \qquad \Rightarrow T_1 = \frac{P_1 V_1}{NK}$$

$$Q_{in} = \frac{V_1}{NK}(P_2 - P_1) C_v$$

• 3 → isobaric

$$P_1 V_1 = NKT_1 \Rightarrow T_1 = \frac{P_1 V_1}{NK}$$

$$P_1 V_2 = NKT_3 \Rightarrow T_3 = \frac{P_1 V_2}{NK}$$

$$Q_{out} = C_p \Delta T = C_p (T_3 - T_1)$$

$$= C_p \frac{P_1}{NK}(V_2 - V_1)$$

• $$W = |Q_{in}| - |Q_{out}| = \frac{1}{NK}\Big[|C_v V_1 (P_2 - P_1)| - |C_p P_1 (V_2 - V_1)| \Big]$$

$$dU = 0$$

$$\Rightarrow Q_{out} + W = Q_{in}$$

$$\Rightarrow W = |Q_{in}| - |Q_{out}|$$

OR

- A → B

$$W = \int P(V)\,dV = \underset{\overset{\parallel}{0}}{Q} - \Delta E_{int} = -\Delta E_{int} = -\frac{f}{2} NK(T_2 - T_3)$$

$$T_3 = \frac{P_1 V_2}{NK} \quad \& \quad T_2 = \frac{P_2 V_1}{NK}$$

$$W = +\frac{3}{2} NK(T_3 - T_2) = \frac{3}{2}(P_1 V_2 - P_2 V_1)$$

- B → C

$$W = \int P\,dV = P_1(V_2 - V_1)$$

- C → A $\quad W = 0$

$$W_{tot} = \frac{3}{2}(P_1 V_2 - P_2 V_1) + P_1(V_2 - V_1)$$

b) $Q_{in} = \frac{V_1}{NK}(P_2-P_1)C_v$

$$Q_{out} = \frac{P_1}{NK}(V_2-V_1)C_p = \frac{C_V}{NK}P_1(V_2-V_1) + P_1(V_2-V_1)$$

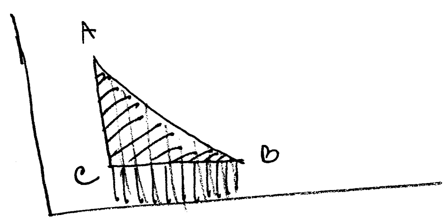
c) $$\eta = 1 - \frac{|Q_{out}|}{|Q_{in}|} = \frac{\frac{V_1}{NK}(P_2-P_1)C_V - P_1(V_2-V_1)\frac{C_V}{NK} - P_1(V_2-V_1)}{\frac{V_1}{NK}(P_2-P_1)C_V}$$

$$= \frac{\frac{C_V}{NK}(P_2V_1 - P_1V_2) - P_1(V_2-V_1)}{\frac{C_V}{NK}(P_2V_1 - P_1V_1)}$$

$$= \frac{P_2V_1 - P_1V_2}{P_2V_1 - P_1V_1} - \frac{NK\, P_1(V_2-V_1)}{C_V(P_2V_1 - P_1V_1)}$$

d) To produce for $W_{\text{by the gas}} > 0$

* $W = \int P\,dV$



* So we need to go clockwise for a pos. work

5. Consider a classical ideal gas in 3D that feels a linear gravitational potential,

$$V(z) = mgz$$

where m is the mass of a single gas atom and $0 < z < \infty$. This is not an interaction between gas atoms, it is simply their gravitational potential energy near the surface of the Earth.

The gas is in a box of dimensions L_x, L_y, and L_z, so that:

$$\begin{aligned} 0 <\ & z < L_z \\ 0 <\ & x < L_x \\ 0 <\ & y < L_y \end{aligned}$$

(a) Calculate the partition function in the canonical ensemble. (3 points)

(b) Determine the internal energy of the gas. (3 points)

(c) Calculate the specific heat c_v. (3 points)

(d) Explain the behavior of the specific heat when $\beta mgL_z >> 1$ and when $\beta mgL_z << 1$. (The approximation for the gravitational potential may or may not be valid for large L_z. Don't worry about that.) (1 point)

5.

a) The one particle Hamiltonian is

sign mistake don't wanna do it again

$$H = \epsilon = \frac{p^2}{2m} \overset{\downarrow}{-} mgz$$

$$= \underbrace{\frac{1}{2m}(P_x^2 + P_y^2 + P_z^2)}_{H_0} \overset{\overset{+}{\downarrow}}{-} \underbrace{mgz}_{H_I}$$

So, the one particle Hamiltonian becomes

$$Z_1 = \frac{1}{(2\pi\hbar)^3}\int d^3x\, d^3p\, e^{-\beta H_0}\, e^{\beta H_I}$$

$$= \frac{L_x L_y 4\pi}{(2\pi\hbar)^3}\int_0^\infty p^2 e^{-\frac{p^2}{2mKT}} dp \int_0^{L_z} dz\, e^{\beta mgz}$$

$$\lambda_T = \sqrt{\frac{2\pi\hbar^2}{mKT}} \qquad = \frac{L_x L_y}{\lambda_T^3}\left(\frac{KT}{mg}\right)\left[e^{\frac{mgL_z}{KT}} - 1\right]$$

$$= \frac{V}{L_z \lambda_T^3}\left(\frac{KT}{mg}\right)\left(e^{\frac{mg}{KT}L_z} - 1\right)$$

So, the N-particle partition func is

$$Z = \frac{1}{N!} Z_1^N$$

(b) $$E = -\frac{\partial}{\partial\beta}\ln Z$$

$$= -N\frac{\partial}{\partial\beta}\ln\left\{\frac{V}{L_z\lambda_T^3}\left(\frac{1}{mg\beta}\right)\left(e^{\beta mgL_z}-1\right)\right\}$$

$$= N\left\{\frac{3}{2}\frac{\partial}{\partial\beta}\ln\beta+\frac{\partial}{\partial\beta}\ln(mg\beta)-\frac{N}{N!}\frac{\partial}{\partial\beta}\ln\left(e^{\beta mgL_z}-1\right)\right\}$$

$$= N\left\{\frac{3}{2\beta}+\frac{mg}{mg\beta}-\frac{N}{N!}\frac{mgL_z\,e^{\beta mgL_z}}{(e^{\beta mgL_z}-1)}\right.$$

$$= N\left(\frac{5}{2\beta}-mgL_z\frac{e^{\beta mgL_z}}{(e^{\beta mgL_z}-1)}\right)$$

(c) $$C_V = \frac{\partial E}{\partial T} = \frac{\partial E}{\partial\beta}\frac{\partial\beta}{\partial T} = \left(-\frac{1}{KT^2}\right)\frac{\partial E}{\partial\beta}$$

$$C_V = -\frac{1}{KT^2}\left\{-\frac{5N}{2\beta^2}-mgL_z\frac{(e^{\beta mgL_z}-1)\,mgL_z e^{\beta mgL_z}-e^{\beta()}mgL_z e^{\beta()}}{(e^{\beta mgL_z}-1)^2}\right\}$$

$$= +\frac{1}{KT^2}\left\{\frac{5N}{2\beta^2}-(mgL_z)^2\frac{e^{\beta mgL_z}}{(e^{\beta mgL_z}-1)^2}\right\}$$

d) $\beta m g L_z >> 1$

$\Rightarrow m g L_z >> KT$

$$C_V = \left\{ \frac{5}{2} NK - \frac{(mgL_z)^2}{KT^2} \frac{1}{1 - e^{-\frac{mgL_z}{KT}}} \right\}$$

For $mgL_z >> KT$

$$C_V \sim \frac{5}{2} NK - \frac{(mgL_z)^2}{KT^2} \quad \text{as} \quad e^{-\frac{mgL_z}{KT}} \to 0$$

So, the gravitational potential energy is large compared to the thermal energy

For $mgL_z << KT$, $e^{-\frac{mgL_z}{KT}} \approx 1 - \frac{mgL_z}{KT}$

$$C_V = \frac{5}{2} NK - \frac{mgL_z}{T}$$

If I fix the sign mistake

I expect

$$C_V \sim \frac{5}{2} NK$$

6. **Boson Magnetism** Consider a gas of non-interacting spin-1 bosons in 3D, each subject to the Hamiltonian

$$\begin{aligned} H(\vec{p}, s_z) &= \frac{p^2}{2m} - \mu_0 s B \\ &= \frac{\hbar^2 k^2}{2m} - \mu_0 s B \end{aligned}$$

where s takes on one of three possible states, $s \in (-1, 0, +1)$, and $\vec{k} \equiv \vec{p}/\hbar$. In this Hamiltonian B is the z-component of the magnetic field, m is the mass of a particle, and μ_0 is the Bohr magneton. (We will ignore the orbital effect (or Lorentz force) where the momentum $\vec{p}$ would have been replaced, $\vec{p} \to \vec{p} + e\vec{A}/c$).

(a) In a grand canonical ensemble of chemical potential μ (which is **not** to be confused with the Bohr magneton, μ_0, above) and temperature T, write down $n_s(\vec{k})$, the average occupation number of the state with wave vector $\vec{k}$ and spin s. (1 point).

(b) Show that the total number of particles in a given spin state s is given by

$$N_s = \frac{V}{\lambda^3} \pi^{3/2} g_{3/2}(z e^{\beta \mu_0 s B})$$

where z is the fugacity, $z = e^{\beta\mu}$, λ is the thermal de Broglie wavelength,

$$\lambda = \frac{h}{\sqrt{2\pi m k_B T}}$$

and $g_p(z)$ is defined on the formula section on page 2 above. (4 points)

(c) The magnetization for fixed μ and T is given by

$$M(T, \mu) = \mu_0 (N_{(+)} - N_{(-)})$$

Show that the zero field susceptibility, χ, is given by:

$$\chi \equiv \left.\frac{\partial M}{\partial B}\right|_{B=0} = \frac{2\mu_0^2}{k_B T} \pi^{3/2} \frac{V}{\lambda^3} g_{1/2}(z).$$

(5 points).

6.

$$H(\bar{p}, s_z) = \frac{\hbar^2 k^2}{2m} - \mu_0 s B \qquad s \in (-1, 0, +1)$$

a)

$$n(\epsilon) = \frac{1}{e^{\beta(\epsilon - \mu)} - 1}$$

$$n_s(\bar{k}) = \frac{1}{e^{\beta\left(\frac{\hbar^2 k^2}{2m} - \mu_0 s B - \mu\right)} - 1}$$

b)

$$N_s = \frac{1}{(2\pi)^3} \int d^3x \, d^3k \; n_s(\bar{k}) \qquad \text{at a fixed } s$$

$$= \frac{V \, 4\pi}{(2\pi)^3} \int_0^\infty dk \frac{k^2}{z^{-1} e^{-\beta \mu_0 s B} \, e^{\beta \frac{\hbar^2 k^2}{2m}} - 1}$$

$$\beta \frac{\hbar^2 k^2}{2m} = x \qquad \Rightarrow k = \sqrt{\frac{2m}{\beta \hbar^2} x}$$

$$\Rightarrow \beta \frac{\hbar^2}{2m} 2k \, dk = dx$$

$$\Rightarrow \quad dk = \frac{m}{\hbar^2 k \beta} dx$$

$$\rightarrow k^2 dk = k \frac{m}{\hbar^2 \beta} dx = \frac{\sqrt{2 m^3 k_B^3 T^3}}{\hbar^3} x^{1/2} \, dx$$

$\frac{3}{2} - 2$
$= -$

$$N_s = \frac{V\,4\pi}{(2\pi)^3}\,\frac{\sqrt{2m^3 k_B^3 T^3}}{\hbar^3}\underbrace{\int_0^\infty \frac{x^{3/2-1}}{\left(z e^{\beta\mu_0 sB}\right)^{-1} e^x - 1}}_{=\,\Gamma(3/2)\,g_{3/2}\left(z e^{\beta\mu_0 sB}\right)}$$

$$= \frac{V\,4\pi}{(2\pi)^3}\,\frac{(2\pi)^3\sqrt{2m^3 k_B^3 T^3}}{h^3} \qquad = \Gamma(3/2)\,g_{3/2}\left(z e^{\beta\mu_0 sB}\right)$$

$$= V\,\frac{\pi^{-3/2}}{\left(\frac{h}{\sqrt{2\pi m k T}}\right)^3}\,\frac{\pi^{1/2}}{2}\,g_{3/2}\left(z e^{\beta\mu_0 sB}\right)$$

$$* \; \Gamma(3/2) = \Gamma\left(\tfrac{1}{2}+1\right) = \tfrac{1}{2}\Gamma\left(\tfrac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$$

$$= \frac{V}{\lambda^3}\,g_{3/2}\left(z e^{\beta\mu_0 sB}\right)$$

c)

$$N_s = \frac{V}{\lambda^3} \; g_{3/2}\left(z\, e^{\beta\mu_0 s B}\right)$$

$$\rightsquigarrow \; g_{3/2}\left(z\, e^{\beta\mu_0 s B}\right) = \frac{z e^{\beta\mu_0 s B}}{1^{3/2}} + \frac{\left(z e^{\beta\mu_0 s B}\right)^2}{2^{3/2}}$$

$$N_+ = \frac{z\, e^{\beta\mu_0 B}}{1^{1/2}} + \frac{e^{2\beta\mu_0 B}}{2} \frac{z^2}{2^{1/2}}$$

$$N_- = -z\, \frac{e^{-\beta\mu_0 B}}{1^{1/2}} - \frac{e^{-2\beta\mu_0 B}}{2} \frac{z^2}{2^{1/2}}$$

$$\rightsquigarrow (N_+ - N_-)^1 = 2z \sinh(\beta\mu_0 B)$$

$$M^1(T,\mu) = 2\mu_0 z \sinh(\beta\mu_0 B)$$

$$\left.\frac{\partial M}{\partial B}\right|^1_{B=0} = 2\mu_0 z \beta\mu_0 \cosh(\beta\mu_0 B)\big|_{B=0}$$

$$= \frac{2\mu_0^2}{k_B T} z$$

$$\rightsquigarrow (N_+ - N_-)^2 = \frac{1}{2} \frac{z^2}{2^{1/2}} \sinh(2\beta\mu_0 B)$$

$$\left.\frac{\partial M}{\partial B}\right|^2_{B=0} = \frac{\mu_0}{2} 4\beta\mu_0 B \frac{z^2}{2^{1/2}} \cosh(2\beta\mu_0 B)\big|_{B=0}$$

$$= \frac{2\mu_0^2}{k_B T} \frac{z^2}{2^{1/2}}$$

* combining the 1st & 2nd order terms

$$\chi = \frac{\partial M}{\partial B}\bigg|_{B=0} = \frac{2\mu_0^2}{K_B T}\frac{V}{\lambda^3}\left(\frac{z^1}{1^{1/2}} + \frac{z^2}{2^{1/2}} + \cdots\right)$$

$$\chi = \frac{2\mu_0^2}{K_B T}\frac{V}{\lambda^3}\, g_{1/2}(z)$$

* retains magnetization

so, <u>ferromagnetic</u>