Classical Mechanics and Statistical/Thermodynamics

January 2007

Possibly Useful Information

Handy Integrals:

$$\int_0^\infty x^n e^{-\alpha x}\, dx = \frac{n!}{\alpha^{n+1}}$$

$$\int_0^\infty e^{-\alpha x^2}\, dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$$

$$\int_0^\infty x\ e^{-\alpha x^2}\, dx = \frac{1}{2\alpha}$$

$$\int_0^\infty x^2\ e^{-\alpha x^2}\, dx = \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}}$$

$$\int_{-\infty}^\infty e^{i a x - b x^2}\, dx = \sqrt{\frac{\pi}{b}} e^{-a^2/4b}$$

Geometric Series:

$$\sum_{n=0}^\infty x^n = \frac{1}{1-x} \qquad \text{for} \quad |x| < 1$$

Stirling's approximation:

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

Riemann and related functions:

$$\sum_{n=1}^\infty \frac{1}{n^p} \equiv \zeta(p)$$

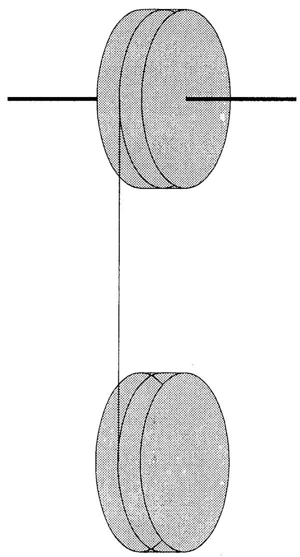
$$\sum_{n=1}^\infty \frac{z^p}{n^p} \equiv g_p(z) \qquad \sum_{n=1}^\infty (-1)^p\ \frac{z^p}{n^p} \equiv f_p(z)$$

$$g_p(1) = \zeta(p) \qquad f_p(1) = \zeta(-p)$$

$$\begin{array}{rr}
\zeta(1) = \infty & \zeta(-1) = 0.0833333 \\
\zeta(2) = 1.64493 & \zeta(-2) = 0 \\
\zeta(3) = 1.20206 & \zeta(-3) = 0.0083333 \\
\zeta(4) = 1.08232 & \zeta(-4) = 0
\end{array}$$

Classical Mechanics

1. A uniform disk of mass M and radius R is attached to a frictionless axle, so that it can spin, but not otherwise move. A string of negligible mass is wrapped around the disk and then wrapped around a second disk also of mass M and radius R. The system starts from rest, and the second disk is released so that it accelerates downward **and** starts to spin.



   (a) Draw a clear, free-body diagram for the system, labelling all forces. Write down Newton's second laws for each part of the system. **(1 point)**

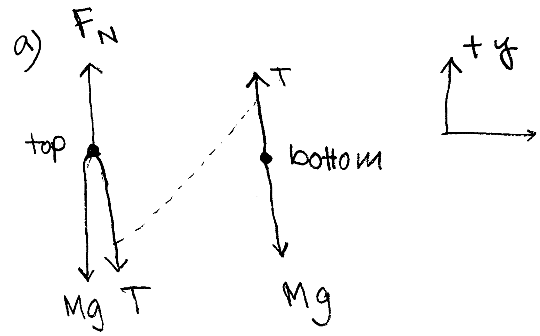
   (b) Solve for the acceleration in the vertical direction of the second disk. **(3 points)**

   (c) Solve for the tension in the string connecting the disks. **(2 points)**

   (d) Using conservation of energy, calculate the vertical speed of the falling disk after it has fallen a distance L. **(4 points)**

1.

a)



b) • bottom

$$T - mg = m\ddot{y}$$

but, for top disk

$$F_N - Mg - T = 0$$

$$\Rightarrow T = F_N - Mg$$

$\xi = R\phi$
$\ddot{\xi} = R\ddot{\phi}$

Also,

$$\underset{\substack{\uparrow \\ \text{cw}}}{\tau_{net} = -TR} = I\alpha = -I\frac{\ddot{y}}{R}$$

$$(mg + m\ddot{y})R = I\frac{\ddot{y}}{R} = -\frac{mR}{2}\ddot{y}$$

* For circular disk $I = \frac{1}{2}mR^2$

$$\Rightarrow \ddot{y}\left(mR + \frac{mR}{2}\right) = -mgR$$

$$\Rightarrow \ddot{y} = -\frac{2}{3}g$$

c) $$T = -\frac{2}{3}mg + mg = \frac{1}{3}mg$$

d) $0 = -mgL + \frac{1}{2}m\dot{y}^2 + \left(\frac{1}{2}I\dot{\theta}^2\right)_{top} + \left(\frac{1}{2}I\dot{\theta}^2\right)_{bottom}$

$\frac{y}{2} = R\theta$ the bottom one covers half a dist'n as the top one

$\dot{y} = 2R\dot{\theta}$ on rolls too at the same rate

$\dot{\theta} = \frac{\dot{y}}{2R}$

$\Rightarrow mgL = \frac{1}{2}m\dot{y}^2 + \frac{1}{2}mR^2\frac{\dot{y}^2}{4R^2}$

* Since I on both top & bottom disk is same & same M & R

$\dot{\theta}_{top} = \dot{\theta}_{bottom}$

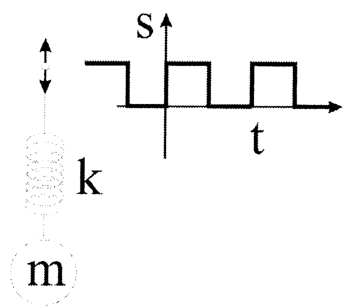
$\Rightarrow mgL = \frac{1}{2}m\dot{y}^2 + \frac{1}{8}m\dot{y}^2$

$\cancel{m}gL = \frac{5}{8}\cancel{m}\dot{y}^2$

$\Rightarrow \dot{y} = \sqrt{\frac{8}{5}gL}$

2. An object of mass, m, hangs on a spring of constant k. The upper end of the spring is moved up and down according to a periodic square wave function as shown. The square wave function may be written:

$$s(t) = 1 \qquad 0 < t < \frac{T}{2}$$
$$s(t) = 0 \qquad \frac{T}{2} < t < T$$



(a) What is the Newtons second law equation for the system? **(2 Points)**

(b) What is the Fourier series respresentation of the driving force for the system? **(3 Points)**

(c) What is the steady state solution for the displacement of the system? **(3 Points)**

(d) If the resonant frequency of the mass-spring system is ω_0 and the period of the driving force equals $\frac{6\pi}{\omega_0}$, what term of the Fourier series will be most important? Why? Consider the case where the damping is small. **(2 Points)**

2.

a) $$V = \Theta(T/2 - t) + \Theta(T - t)$$

3. Consider a non-relativistic charged particle moving in an arbitrary time-independent electric and magnetic field with electric potential $\Phi(\vec{r})$ and vector potential $\vec{\mathcal{A}}(\vec{r})$. The Lagrangian for this system is

$$L = T - U = \frac{1}{2}m\vec{v}^2 - q\,\Phi + \frac{q}{c}\,\vec{v}\cdot\vec{\mathcal{A}}$$

   (a) Prove that the equations of motion:

$$\vec{F} = q\left(\vec{\mathcal{E}} + \frac{\vec{v}}{c}\times\vec{\mathcal{B}}\right),$$

   where $\vec{\mathcal{E}}$ and $\vec{\mathcal{B}}$ are the electric and magnetic fields, follow from the Lagrangian.

   (b) Find the Hamiltonian for this system.

   (c) Now assume that $\vec{\mathcal{E}} = \mathcal{E}_0\,\hat{\imath}$ and $\vec{\mathcal{B}} = \mathcal{B}_0\,\hat{k}$ are uniform, constant and perpendicular. Assuming that $\vec{r}(0) = 0$, solve for the trajectory $\vec{r}(t)$.

3.

a) $L = \frac{1}{2} m \dot{r}^2 - q\Phi(r) + \frac{q}{c} \dot{r} \cdot \bar{A}$

$$\dot{\left(\frac{\partial L}{\partial \dot{r}}\right)} - \frac{\partial L}{\partial r} = 0$$

$$\Rightarrow \quad m\ddot{r} + \frac{q}{c} \cancelto{0}{\frac{dA}{dt}} + q \frac{\partial \Phi}{\partial r} - \frac{q}{c} \dot{r} \cdot \frac{\partial \bar{A}}{\partial r} = 0$$

$$\Rightarrow \quad m\ddot{r} = \underbrace{-q \frac{\partial \Phi}{\partial r}}_{= q\mathcal{E}} + \frac{q}{c} \dot{r} \cdot \frac{\partial \bar{A}}{\partial r}$$

NOW,
$$\dot{\bar{r}} \times \bar{B} = \dot{\bar{r}} \times (\nabla \times \bar{A}) = \bar{\nabla}(\dot{\bar{r}} \cdot \bar{A}) - \bar{A} \underbrace{(\bar{\nabla} \cdot \dot{\bar{r}})}_{= 0}$$

$$= \dot{\bar{r}}(\bar{\nabla} \cdot \bar{A})$$

$$= \dot{\bar{r}} \cdot \frac{\partial \bar{A}}{\partial r}$$

$$\Rightarrow \quad \bar{F} = q\left(\bar{\mathcal{E}} + \frac{1}{c} \dot{\bar{r}} \wedge \bar{B}\right)$$

b) $$\bar{P}_r = \frac{\partial L}{\partial \dot{\bar{r}}} = m\dot{\bar{r}} + \frac{q}{c}\bar{A}$$

$$\dot{\bar{r}} = \frac{\left(\bar{P}_r - \frac{q}{c}\bar{A}\right)}{m}$$

$$H = \bar{P}_r \dot{\bar{r}} - L$$

$$= \frac{1}{2} m \dot{\bar{r}} \cdot \dot{\bar{r}} + q\,\Phi(r)$$

$$= \frac{\left(\bar{P}_r - \frac{q}{c}\bar{A}\right)^2}{2m} + q\,\Phi(r)$$

c) $$\dot{r} = \frac{\partial H}{\partial P_r} = \frac{\left(P_r - \frac{q}{c}\bar{A}\right)}{m}$$

$$\dot{P}_r = -\frac{\partial H}{\partial r} = -\frac{P_r - \frac{q}{c}\bar{A}}{m}\left(-\frac{q}{c}\frac{\partial \bar{A}}{\partial r}\right) - q\frac{\partial \bar{\Phi}(r)}{\partial r}$$

$$= \frac{q}{m}\left(P_r - \frac{q}{c}\bar{A}\right)\left(B_0 \hat{k}\right) + q\,\mathcal{E}_0\,\hat{i}$$

c)

$$m\ddot{\bar{r}} = q\bar{\mathcal{E}} + \frac{q}{c}\dot{\bar{r}}\wedge\bar{B}$$

$$= q\mathcal{E}_0\hat{x} + \frac{q}{c}\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{pmatrix}\wedge\begin{pmatrix}0\\ B_0\\ 0\end{pmatrix}$$

$$= q\mathcal{E}_0\hat{x} + \frac{q}{c}\begin{pmatrix}-\dot{z}B_0\\ 0\\ \dot{x}B_0\end{pmatrix}$$

$$= q\begin{pmatrix}\mathcal{E}_0 - \dot{z}\frac{B_0}{c}\\ 0\\ \dot{x}\frac{B_0}{c}\end{pmatrix}$$

(1) $\ddot{x} + \dot{z}\frac{qB_0}{mc} = \underbrace{\frac{q\mathcal{E}_0}{m}}_{\beta}$

(3) $\Rightarrow \ddot{x} + \alpha^2 x = \beta$

$\rightsquigarrow \ddot{x} + \alpha^2 x = 0$

$\lambda = \pm i\alpha$

$x_c(t) = G\cos\alpha t + H\sin\alpha t$

& (2) $\ddot{z} - \dot{x}\overbrace{\frac{qB_0}{mc}}^{\alpha} = 0 \Rightarrow \dot{x} = \frac{\ddot{z}}{\alpha}$

$\cancel{\frac{d\dot{z}}{dz}\frac{dz}{dt} = \dot{x}\alpha}$

$\cancel{\Rightarrow \dot{z}\,d\dot{z} = \dot{x}\alpha\,dz}$

$\cancel{\Rightarrow \frac{\dot{z}^2}{2} = \dot{x}\alpha z}$

$\cancel{\Rightarrow \dot{z} = \dot{x}^{1/2}\sqrt{\alpha z}}$

From (1)

$\Rightarrow \frac{dx}{dz}\dot{z} = \frac{\dot{z}}{\alpha}\frac{d\dot{z}}{dz}$

$\Rightarrow dx = \frac{1}{\alpha}d\dot{z}$

$\Rightarrow x = \frac{\dot{z}}{\alpha}$

$\Rightarrow \dot{z} = \alpha x$ (3)

$\rightsquigarrow$ let $x_p(t) = Dt^2 + Et + F$

$\ddot{x}_p = 2Dt$

then $2Dt + \alpha(Dt^2 + Et + F) = \beta$

$\therefore D = 0 \quad F = \beta/\alpha$

$E = 0$

$x_p(t) = \beta/\alpha = \frac{c\mathcal{E}_0}{B}$

thus,

$$X(t) = G\cos\alpha t + H\sin\alpha t + \frac{c\mathcal{E}_0}{B}$$

At $t=0$ $\quad X(t)=0$

$$\Rightarrow G = -\frac{c\mathcal{E}_0}{B}$$

$$X(t) = \frac{c\mathcal{E}_0}{B}(1-\cos\alpha t) + H\sin\alpha t$$

Now,

$$\dot{z} = \alpha X$$

$$z(t) = \frac{\alpha c\mathcal{E}_0}{B}t - \frac{\cancel{\alpha} c\mathcal{E}_0}{\cancel{\alpha} B}\sin\alpha t - H\cos\alpha t$$

At $t=0$, $z(0)=0$, $H=0$

Thus,

$$X(t) = \frac{c\mathcal{E}_0}{B_0}(1-\cos\alpha t)$$

$$z(t) = \frac{c\mathcal{E}_0}{B_0}\left(\frac{qB_0 t}{mc} - \sin\alpha t\right)$$

$$\bar{r}(t) = \frac{c\mathcal{E}_0}{B_0}\left[\hat{x}\left(1-\cos\left(\frac{qB_0}{mc}\right)t\right) + \hat{y}\left(\frac{qB_0}{mc}t - \sin\left(\frac{qB_0}{mc}\right)t\right)\right]$$

Statistical Mechanics

4. A certain system can be modelled as an ideal gas of point particles, but the point particles have two internal states, with energies 0 and Δ.

   (a) Show that in the canonical ensemble the partition function $Z(T, V, N)$ for the gas can be written as

   $$Z(T, V, N) = Z_0 \left(1 + e^{-\Delta/kT}\right)^N \frac{\left(VT^{3/2}\right)^N}{N!}$$

   where Z_0 is a multiplicative constant that has no effect on the equation of state. **(2 points)**

   (b) Calculate the specific heat at constant volume for the gas.

   (c) Assume further that we have *two* such gases, A and B, and that each has an internal state, but that $\Delta_A \neq \Delta_B$. Determine $Z(T, V, N_A, N_B)$, where N_A and N_B are the number of gas atoms of type A and B, respectively. **(1 point)**

   (d) Finally, if gas particles of type A can convert into type B and vice versa, calculate N_A/N_{tot} in equilibrium, where $N_{\text{tot}} = N_A + N_B$. **(5 point)**

4.

The one particle Hamiltonian can be written as,

$$H = \underbrace{\frac{p^2}{2m}}_{H_0} + \underbrace{\epsilon}_{H_1} \qquad \text{where,} \quad \epsilon = 0 \text{ or } \Delta$$

The one particle partition func is

$$Z_1 = Z_0' \, Z_I'$$

$$Z_0' = \frac{1}{(2\pi\hbar)^3} \int d^3x \, d^3p \; e^{-\beta H_0}$$

$$= \frac{V}{(2\pi\hbar)^3} \, 4\pi \int_0^\infty dp \; p^2 \, e^{-\frac{p^2}{2mKT}}$$

$$= \frac{V}{(2\pi\hbar)^3} \, 4\pi \, \frac{1}{4} \sqrt{\pi (2mKT)^3}$$

$$= \frac{\pi^{3/2} V (2mK)^{3/2} T^{3/2}}{(2\pi\hbar)^3}$$

$$Z_1' = \sum_{\epsilon_i} e^{-\beta\epsilon_i} = 1 + e^{-\beta\Delta}$$

Thus,

$$Z_1 = \frac{\pi^{3/2}(2mK)^{3/2}}{(2\pi\hbar)^3} V T^{3/2} \left(1 + e^{-\Delta/KT}\right)$$

* Since they are non-interaction, linearly independent & indistinguishable,

The total partition func is

$$Z = \frac{1}{N!} Z_1^N$$

$$= \frac{1}{N!} \underbrace{\left(\frac{\pi^{3/2} (2mK)^{3/2}}{(2\pi\hbar)^3} \right)^N}_{Z_0} \left(V T^{3/2} \right)^N \left(1 + e^{-\Delta/KT} \right)^N$$

$$= Z_0 \left(1 + e^{-\Delta/KT} \right)^N \frac{\left(V T^{3/2} \right)^N}{N!}$$

(b) $$E = -\frac{\partial}{\partial \beta} \ln Z$$

$$= -N \frac{\partial}{\partial \beta} \ln \left(1 + e^{-\beta\Delta} \right) + \frac{3N}{2} \frac{\partial}{\partial \beta} \ln \left(K\beta \right)$$

$$= -N \frac{(-\Delta) e^{-\beta\Delta}}{1 + e^{-\beta\Delta}} + \frac{3N}{2} \frac{K}{K\beta}$$

$$= N\Delta \frac{e^{-\beta\Delta}}{1 + e^{-\beta\Delta}} + \frac{3N}{2\beta}$$

$$C_V = \frac{\partial E}{\partial T} = \frac{\partial E}{\partial \beta}\frac{\partial \beta}{\partial T}$$

$$= \left(-\frac{1}{KT^2}\right)\frac{\partial}{\partial \beta}\left\{ N\Delta \frac{e^{-\beta\Delta}}{1+e^{-\beta\Delta}} + \frac{3N}{2\beta}\right\}$$

$$= \left(-\frac{1}{KT^2}\right)\left[N\Delta\left\{\frac{(-\Delta)\,e^{-\beta\Delta}}{1+e^{-\beta\Delta}} - \frac{e^{-\beta\Delta}}{(1+e^{-\beta\Delta})^2}\right\} - \frac{3N}{2\beta^2}\right]$$

$$= \frac{N\Delta^2 k\beta^2 e^{-\beta\Delta}}{(1+e^{-\beta\Delta})} - \frac{N\Delta k\beta^2 e^{-\beta\Delta}}{(1+e^{-\beta\Delta})^2} + \frac{3N}{2}$$

$$= \frac{NK\beta^2 e^{-\beta\Delta}}{(1+e^{-\beta\Delta})}\left\{\Delta^2 - \frac{\Delta}{1+e^{-\beta\Delta}}\right\} + \frac{3N}{2}$$

c) $$Z(T, V, N_A, N_B) = Z_A\ Z_B$$

$$= Z_0^2\left(1+e^{-\Delta_A/KT}\right)^{N_A}\left(1+e^{-\Delta_B/KT}\right)^{N_B}\frac{(VT^{3/2})^{N_A}}{N_A!}\frac{(VT^{3/2})^{N_B}}{N_B!}$$

d) The Free energy

$$F = -KT \ln Z(T, V, N_A, N_B)$$

$$= -N_A KT \ln\left(1+e^{-\frac{\Delta_A}{KT}}\right) - N_B KT \ln\left(1+e^{-\frac{\Delta_B}{KT}}\right)$$

$$- \frac{3 N_A KT}{2} \ln T \overset{*KT}{+} (N_A \ln N_A - N_A) - \frac{3N_B}{2} KT$$

$$+ (N_B \ln N_B - N_B) KT$$

At equilibrium, $\mu_A = \mu_B$

$$\mu = \left.\frac{\partial F}{\partial N}\right|_{V,T}$$

$$\mu_A = \left.\frac{\partial F}{\partial N_A}\right|_{V,T}$$

$$= -KT \ln\left(1+e^{-\Delta/KT}\right) - \frac{3}{2} KT \ln(T) + KT \ln N_A$$

$$= KT \ln\left(\frac{N_A}{T\left(1+e^{-\Delta/KT}\right)}\right)$$

Similarly,

$$\mu_B = KT \ln\left(\frac{N_B}{T\left(1+e^{-\Delta/KT}\right)}\right)$$

At equilibrium

$$* \quad \frac{N_A}{N_{tot}} = \left(\frac{N_{tot}}{N_A} = 1 + \frac{N_B}{N_A} \right)$$

$$\frac{N_A}{1+e^{-\beta\Delta_A}} = \frac{N_B}{1+e^{-\beta\Delta_B}}$$

$$\Rightarrow \quad \frac{N_B}{N_A} = \frac{1+e^{-\beta\Delta_B}}{1+e^{-\beta\Delta_A}}$$

Thus,

$$\frac{N_{tot}}{N_A} = 1 + \frac{1+e^{-\beta\Delta_B}}{1+e^{-\beta\Delta_A}}$$

$$= \frac{1+e^{-\beta\Delta_A}+1+e^{-\beta\Delta_B}}{1+e^{-\beta\Delta_A}}$$

$$\therefore \quad \frac{N_A}{N_{tot}} = \frac{1+e^{-\beta\Delta_A}}{2+e^{-\beta(\Delta_A+\Delta_B)}}$$

5. A gas of N distinguishable classical non-interacting atoms is held in a neutral atom trap by a potential of the form $V(\vec{r}) = a\,r$ where $r = \sqrt{x^2 + y^2 + z^2}$. The gas is in thermal equilibrium at a temperature T.

   (a) Find the single particle partition function Z_1 for a trapped atom. Express your answer in the form $Z_1 = A\,T^{\alpha} a^{-\eta}$. Find the prefactor A and the exponents α and η. **(3 points)**

   (b) Find the entropy of the gas in terms of N, k, and $Z_1(T, a)$. Do not leave any derivatives in your answer. **(4 points)**

   (c) The gas can be cooled if the potential is lowered reversibly (by decreasing a) while no heat is allowed to be exchanged with the surroundings, $dQ = 0$. Under these conditions, find T as a function of a and the initial values T_0 and a_0. **(4 points)**

5.

a) The one particle Hamiltonian is,

$$H = \frac{p^2}{2m} + V(r)$$

$$= \frac{p^2}{2m} + ar$$

So, the one particle partition func

$$Z_1 = \frac{1}{(2\pi\hbar)^3} \int d^3x \, d^3p \; e^{-\beta H}$$

$$= \frac{1}{(2\pi\hbar)^3} \int r^2 \sin\theta \, d\theta \, d\phi \, dr \; e^{-ar} \int p^2 \sin\theta_p \, d\theta_p \, dp \; e^{-p^2/2mKT}$$

$$= \frac{4\pi \, 4\pi}{(2\pi\hbar)^3} \int_0^\infty r^2 e^{-\frac{ar}{KT}} dr \int_0^\infty dp \; p^2 \, e^{-p^2/2mKT}$$

$$= \frac{(4\pi)^2}{(2\pi\hbar)^3} \; \frac{2!}{(a/KT)^3} \; \frac{1}{4}\sqrt{\pi (2mKT)^3}$$

$$= \frac{8\pi^3 K^3}{(2\pi\hbar)^3} \; a^{-3} \; T^{+9/2} \; (2mK)^{3/2}$$

$$= \left(\frac{8m^3K^{9/2}}{\hbar^3}\right) T^{+9/2} \; a^{-3}$$

Thus,

$$A = \left(\frac{2mk^{3/2}}{\hbar}\right)^3$$

$$\alpha = 9/2$$

$$\eta = 3$$

b) $$S = -\left.\frac{\partial F}{\partial T}\right|_{V,N}$$

$$= \frac{\partial}{\partial T}(kT \ln Z) = \frac{\partial}{\partial T}(NkT \ln Z_1)$$

but $Z = Z_1^N = A^N T^{9N/2} a^{-3N}$

$$Z_1 = A T^{9/2} a^{-3}$$

$$S = \frac{\partial}{\partial T}\left(NkT \ln Z_1\right)$$

$$= Nk \ln Z_1 + NkT \cdot \frac{\partial \ln Z_1}{\partial T}$$

$$= Nk \ln Z_1 + NkT \frac{\partial}{\partial T}\left(\frac{9}{2}\ln T\right)$$

$$= Nk \ln Z_1 + \frac{9Nk}{2}$$

c)

$$F = E - TS \qquad \text{for } đQ = 0$$

$$dF = -PdV - SdT - TdS + \mu dN$$

$$T = -\frac{\partial F}{\partial S}\Big|_{V,T,N}$$

$$= \left(-\frac{\partial F}{\partial T}\frac{\partial T}{\partial S}\right)_{V,T,N}$$

not sure if my def is right

but, $S\,(\text{at } T = T_o) = -\frac{\partial F}{\partial T}\Big|_{V,N,T=T_o}$

$$= \underline{S(T_o)}\left[\frac{\partial S}{\partial T}\right]^{-1}_{V,N,T=T_o}$$

$$\frac{\partial S}{\partial T}\Big|_{T=T_o} = NK\frac{\partial}{\partial T}(\ln Z_1)$$

$$= NK\frac{9}{2T}$$

$$\left[\frac{\partial S}{\partial T}\right]^{-1}_{V,N,T=T_o} = \frac{2T_o}{9NK}$$

$$T = \frac{2T_o}{9NK}\left\{NK\ln\left(A\,T_o^{9/2}a^{-3/2}\right) + \frac{9NK}{2}\right\}$$

At, $T = T_0$, $a = a_0$

$$\frac{9NK}{2} = NK \ln\left(A T_0^{9/2} a_0^{-3/2}\right) + \frac{9NK}{2}$$

$$\ln\left(A T_0^{9/2}, a_0^{-3/2}\right) = \ln 1$$

$$\Rightarrow \quad A T_0^{9/2} a_0^{-3/2} = 1$$

$$\Rightarrow \quad a_0^{-3/2} = \frac{1}{A T_0^{9/2}}$$

Hence,

$$T = \frac{2T_0}{9NK} \left\{ NK \ln\left(a_0^{3/2} a^{-3/2}\right) + \frac{9NK}{2} \right\}$$

$$= \frac{2}{9} T_0 \ln\left(\frac{a_0}{a}\right)^{3/2} + T_0$$

c) For adiabatic process $dQ = 0$

So, $dS = 0$

$$dS = d\left[NK\ln\left(AT^{9/2}a^{-3}\right) + \frac{9NK}{2}\right] = 0$$

$$\Rightarrow \frac{NK}{AT^{9/2}a^{-3}}\left(\frac{9}{2}a^{-3}AT^{+7/2}dT - 3AT^{9/2}a^{-4}da\right) = 0$$

$$\Rightarrow A\frac{9}{2}T^{-1}dT - 3Aa^{-1}da = 0$$

$$\Rightarrow \frac{3}{2}\int_{T_0}^{T}\frac{dT}{T} = \int_{a_0}^{a}\frac{da}{a}$$

$$\Rightarrow \ln\left(\frac{T}{T_0}\right) = \ln\left(\frac{a}{a_0}\right)^{3/2}$$

$$\Rightarrow T = T_0\left(\frac{a}{a_0}\right)^{3/2}$$

6. Consider a fictitious spin 5/2 fermion with the charge of an electron but with a dispersion relationship

$$E = v_0 p.$$

where $p \equiv |\vec{p}|$. We will call this particle the "offon." Assume that your offons are confined in a three dimensional sample and are non-ineracting. We will work in the Grand Canonical Ensemble.

   (a) Determine the density, $\rho = \langle N \rangle / V$, as a function of the chemical potential μ (or the fugacity, $z \equiv e^{\beta\mu}$), T, and V. **(3 points)**

   (b) What is the offonic Fermi energy (μ at T =0) as a function of their density? (*Hint:* This should not involve any complicated integrals). **(3 points)**

   (c) Derive a series expansion in z for the grand canonical free entropy, $\Xi = \frac{PV}{kT} = \log \mathcal{Z}$, where $\mathcal{Z}$ is the grand canonical partition function. **(4 points)**