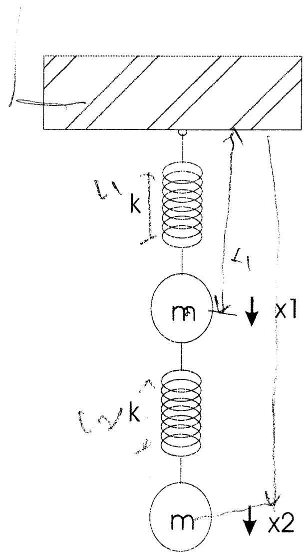
Mechanics and Statistical Mechanics Qualifying Exam
Spring 2006

5-06

Problem 1: (10 Points)

Identical objects of equal mass, m, are hung on identical springs of constant k. When these objects are displaced by distances x_1 and x_2 *from their equilibrium position* the resulting forces give rise to oscillations. Neglect damping and assume the motion is in 1-D along the x displacement.



a. What are the normal mode oscillations that you expect to see? Just give the number of normal modes and the relative directions of the oscillations of the two masses for each normal mode. **(1 Points)**

b. What are the equations of motion that describe the oscillations of m_1 and m_2? **(2 Points)**

c. What are the normal mode frequencies? **(4 Points)**

d. What are the ratios of the amplitudes for the displacement x_1 and x_2 for each of the normal modes? **(2 Points)**

e. Which of the normal mode frequencies corresponds to which of the normal modes you wrote down in (a.)? **(1 Point)**

* x_1 & x_2 usual co-ordinate
* $\eta_1 \eta_2$ small co-ordinate

1.

$$L = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2 - \frac{1}{2} K (x_1 - L_1)^2 - \frac{1}{2} K (x_2 - x_1 - L_1)^2$$

$$+ mgx_1 + mgx_2$$

$$\eta_1 = x_1 - L_1 \qquad \Rightarrow \dot{\eta}_1 = \dot{x}_1$$

$$\eta_2 = x_2 - L_1 - L_2 \quad \Rightarrow \dot{\eta}_2 = \dot{x}_2$$

$$L = \frac{1}{2} m \dot{\eta}_1^2 + \frac{1}{2} m \dot{\eta}_2^2 - \frac{1}{2} K \eta_1^2 - \frac{1}{2} K \eta_2^2 + mg\eta_1 + mg\eta_2$$

$$+ mg(L_1 + L_2)$$

$$V_{eff} = \frac{1}{2} \{ K\eta_1^2 + K\eta_2^2 - mg\eta_1 - mg\eta_2 \}$$

$$\frac{\partial V_{eff}}{\partial \eta_1} = \frac{1}{2} \{ 2K\eta_1 - mg \} \qquad \tilde{T} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$$

$$\frac{\partial^2 V_{eff}}{\partial \eta_1^2} = \frac{1}{2} \{ 2K \} \Big|_{\eta_1 = 0} \qquad \tilde{V} = \begin{pmatrix} +2K & 0 \\ 0 & +2K \end{pmatrix}$$

$$\frac{\partial^2 V_{eff}}{\partial \eta_2^2} = \frac{1}{2} \{ 2K \}$$

Problem 2: (10 Points)

A rocket with initial mass m_0 takes off from the surface of the earth. Fuel is ejected at a constant rate, β, with a velocity u relative to the rocket. Neglect air resistance.

a. By considering momentum conservation, p(t+dt)-p(t)=0, derive a relation between the change in mass, dm, and the change in velocity, dv, of the rocket. (Hint: your expression should contain the mass, m, and the velocity of the fuel, u. Neglect terms that are second order in dm and dv.) **(3 Points)**

b. Derive a differential equation for the acceleration of the rocket. **(2 Points)**

c. What is the minimum burn rate for the fuel so that the rocket makes it off of the ground and what is the burn rate if the initial acceleration is 4g? **(2 Points)**

d. Find dv/dt as a function of time. Assume that you can treat the gravitational force as constant. (Hint: remember that the mass is a function of time.) **(2 Points)**

e. Solve for v(t). **(1 Points)**

2.

a) * Momentum conservation,

dm' = fuel mass
u = fuel velocity

$$P(t) = p(t+dt)$$

$$\Rightarrow mV = (m - dm')(v+dv) + dm'(v-u)$$

$$\Rightarrow \cancel{mV} = \cancel{mV} + mdV - V\cancel{dm'} + V\cancel{dm'} - udm' + \partial^2$$

$$\Rightarrow 0 = mdV - udm'$$

but $dm' = -dm$
↑ mass lost for the rocket

$$\Rightarrow 0 = mdV + udm$$

b) $p(t+dt) - p(t) = dp$

$$\Rightarrow mdV + udm = dp = Fdt$$

$$\Rightarrow m\dot{v} + u\dot{m} = -mg$$

(u ↑ const)

$$\Rightarrow a = -g - \frac{u}{m}\frac{dm}{dt}$$

c) burn rate $\beta = \dfrac{dm'}{dt} = -\dfrac{dm}{dt}$

$$a = -g + \frac{u}{m}\beta$$

For just lift off the ground, $a = 0$?

$$\beta_{min} = \frac{mg}{u}$$

if $a = 4g$

$$4g = -g + \frac{u}{m}\beta$$

$$\Rightarrow \beta = \frac{5mg}{u}$$

d)

$$\frac{dv}{dt} = -g + \frac{u}{m(t)}\beta$$

Now,

$$\beta = -\frac{dm}{dt}$$

$$\Rightarrow \int_{m_0}^{m} dm = -\beta\int_0^t dt$$

$$= m(t) = m_0 - \beta t$$

$$\Rightarrow \frac{dv}{dt} = -g + \frac{u}{m_0 - \beta t}\beta$$

$$\Rightarrow \int_0^v dv = -gt + u\beta\underbrace{\int_0^t \frac{dt}{m_0 - \beta t}}$$

$m_0 - \beta t = x$

$dt = -\frac{1}{\beta}dx$

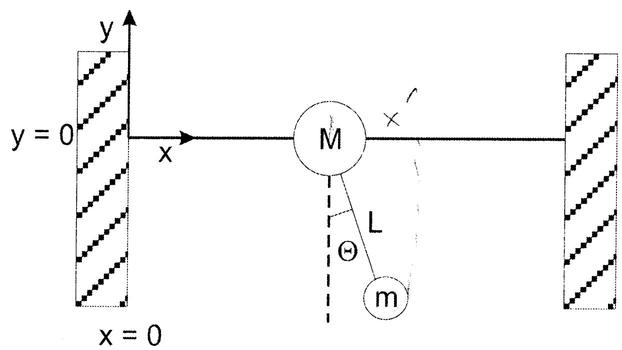
$t = 0,\ x = m_0$

$$= -u\ln\left(\frac{m_0 - \beta t}{m_0}\right)$$

$$\Rightarrow v(t) = -gt - u\ln\left(\frac{m_0 - \beta t}{m_0}\right)$$

Problem 3 (10 Points):

Consider a mass m on a string of fixed length L in a uniform gravitational field. The upper end of the string is attached to a mass M which can move in the horizontal direction. Consider only planar motion as shown in the figure. Use the generalized coordinates x and Θ to answer the following questions.



a. Find the Lagrangian for the system. **(3 Points)**

b. Find Lagrange's equations of motion. **(2 Points)**

c. Is $H = T + U$? You must explain your answer in words to receive any credit. **(2 Points)**

d. For the motion in Θ, assume small oscillations and linearize the equation. Consider the limit $M \gg m$. Explain the nature of the motion in this limit both mathematically and physically. **(3 Points)**

3.

a) $L = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}_1^2 + \dot{y}_1^2) - mgy_1$

$$x_1 = x + L\sin\theta$$

$$\dot{x}_1 = \dot{x} + L\cos\theta\,\dot{\theta}$$

$$y_1 = -L\cos\theta \Rightarrow \dot{y}_1 = L\sin\theta\,\dot{\theta}$$

$$L = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x} + L\cos\theta\,\dot{\theta})^2 + \frac{1}{2}m(L\sin\theta\,\dot{\theta})^2 + mgL\cos\theta$$

$$L = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\dot{x}^2 + mL\cos\theta\,\dot{\theta}\dot{x} + \frac{1}{2}mL^2\dot{\theta}^2 + mgL\cos\theta$$

b) $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$

$$\Rightarrow (M+m)\ddot{x} + mL\cos\theta\,\ddot{\theta} - mL\sin\theta\,\dot{\theta}^2 = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\Rightarrow mL\cos\theta\,\ddot{x} - mL\sin\theta\,\dot{\theta}^2 + mL^2\ddot{\theta} - mgL\sin\theta = 0$$

c. The Lagrangian is independent of time and so is all the co-ordinate transformations,

Hence,

$$H = T + U = E, \text{ the total energy}$$

d.

$$\dot{P}_x = 0$$

$$P_x = M\dot{x} + m\dot{x} + mL\cos\theta\,\dot{\theta}$$

$$\Rightarrow \quad \dot{x} = \frac{P_x - mL\cos\theta\,\dot{\theta}}{(M+m)}$$

$$R = L - P_x\dot{x}$$

$$= -\frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}mL^2\dot{\theta}^2 + mgL\cos\theta$$

$$= -\frac{(P_x - mL\cos\theta\dot{\theta})^2}{2(M+m)} + \frac{1}{2}mL^2\dot{\theta}^2 + mgL\cos\theta$$

$$= -\frac{P_x^2}{2(M+m)} + \frac{mL}{(M+m)}P_x\cos\theta\,\dot{\theta} + \frac{m^2L^2\cos^2\theta}{2(M+m)}\dot{\theta}^2$$

$$+ \frac{1}{2}mL^2\dot{\theta}^2 + mgL\cos\theta$$

why this way doesn't work?

c) For small oscillation,

$$T = \frac{1}{2} M\dot{x}^2 + \frac{1}{2} m\dot{x}^2 + mL\dot{\theta}\dot{x} + mL^2\dot{\theta}^2$$

as $\cos\theta = 1 - \frac{\theta^2}{2}$ (↑ at $\theta = 0$)

$$\tilde{T} = \begin{pmatrix} M+m & mL \\ mL & mL^2 \end{pmatrix}$$

$$V = -mgL\cos\theta$$

$$\approx + \frac{mgL\theta^2}{2}$$

$$\cos\theta = 1 - \frac{1}{2!}\frac{\partial^2 \cos\theta}{\partial\theta^2}\Big|_{\theta=0}\theta^2$$

$$= -\frac{\theta^2}{2}$$

$$\tilde{V} = \begin{pmatrix} 0 & 0 \\ 0 & mgL \end{pmatrix}$$

$$|\tilde{V} - \lambda\tilde{T}| = 0$$

$$\Rightarrow \begin{pmatrix} -\lambda(M+m) & -\lambda mL \\ -\lambda mL & mgL - \lambda mL^2 \end{pmatrix} = 0$$

$$\Rightarrow -\lambda(M+m)\,mL\,(g - \lambda L) - \lambda^2 m^2 L^2 = 0$$

$$\Rightarrow \lambda mL\left[(M+m)(g-\lambda L) + \lambda mL\right] = 0$$

$$\lambda mL + (M+m)g - \lambda(M+m)L = 0$$

$$\Rightarrow -\lambda LM + (M+m)g = 0$$

$$\Rightarrow \lambda = \frac{(M+m)g}{LM}$$

$$\Rightarrow \omega = \sqrt{\frac{(M+m)g}{ML}}$$

for $M \gg m$

$$\omega = \Big(1+\underbrace{\frac{m}{M}}_{\ll 1}\Big)^{1/2}\sqrt{\frac{g}{L}}$$

$$\approx \Big(1+\frac{m}{2M}\Big)\sqrt{g/L}$$

If M is heavier enough than m, taking the leading order would give,

$\omega = \sqrt{g/L}$ which is the oscillation of a SHO fixed at a pt.

S-06

Problem 4 (10 Points):

This problem considers a photon gas. A blackbody cavity can be considered to contain a gas that obeys the equations of state:

$$U = b\, V\, T^4$$

$$P\, V = \frac{1}{3}\, U$$

where U is the internal energy, V is the volume, P is the pressure, T is the temperature and $b = 7.56 \times 10^{-16}\, \mathrm{J/(m^3\, K^4)}$. Note that there is no dependence on N, the number of particles.

a. Show that the fundamental equation for the entropy, S(U,V) is:

$$S(U,V) = \frac{4}{3}\, b^{1/4}\, U^{3/4}\, V^{1/4}$$

adiabatic + quasistatic

(3 Points)

b. The universe can be treated as an expanding electromagnetic cavity at a temperature of $T = 2.7\,\mathrm{K}$. Assume the expansion of the universe is isoentropic. What will the temperature of the universe be when it is twice its current size? **(2 Points)**

$Q=0$ $\Delta S=0$

c. What is the pressure associated with the electromagnetic radiation? **(1 Points)**

d. What is the Helmholtz potential for this system as a function of U and P? **(3 Points)**

e. Why is there no dependence upon N in the fundamental equation for the photon gas? **(1 Points)**

Have to know it in more detail

4.

a) $U = bVT^4$

$dU = TdS - PdV + \sum_i \mu_i dN_i$

$\Rightarrow \left.\frac{dS}{dU}\right|_{V,N} = \frac{1}{T} \Rightarrow dS = \frac{1}{T}dU = \left(\frac{U}{bV}\right)^{1/4} dU$

$\Rightarrow S = (bV)^{1/4} \int U^{-1/4} dU = (bV)^{1/4} U^{3/4} \frac{4}{3}$

$S(U,V) = \frac{4}{3} b^{1/4} U^{3/4} V^{1/4}$

b) * isoentropic = adiabatic + quasi-static ; $Q=0$ & $\Delta S=0$

So entropy is const.

$S = \frac{4}{3} b^{1/4} U^{3/4} V^{1/4} = \text{const.}$

$S = \frac{4}{3} b^{1/4} b^{3/4} V^{3/4} T^3 V^{1/4} = \text{const}$

$\Rightarrow \frac{4}{3} bVT^3 = \text{const}$

Thus, $V_1 T_1^3 = V_2 T_2^3$

$\Rightarrow T_2 = \left(\frac{V_1 T_1^3}{V_2}\right)^{1/3}$

but $V_2 = 2V_1$

$$T_2 = \left(\tfrac{1}{2} T_1^3\right)^{1/3} = 2^{-1/3} T_1$$

$$T_2 = 2^{-1/3} T_1$$

c. $$PV = \tfrac{1}{3} U \Rightarrow P = \tfrac{1}{3} \frac{U}{V} = \tfrac{1}{3} b T^4$$

d. The Helmholtz potential,

$$\frac{U^{1/4}}{b^{1/4}V^{1/4}} = \frac{U^{1/4} 3^{1/4} P^{1/4}}{b^{1/4} U^{1/4}} = 3^{+1/4} P^{1/4} b^{-1/4}$$

$$F = U - TS$$

$$= U - T\left(\tfrac{4}{3} b^{1/4} U^{3/4} V^{1/4}\right)$$

$$V^{1/4} = \frac{U^{1/4}}{3^{1/4} P^{1/4}}$$

$$= U - T \tfrac{4}{3} 3^{-1/4} b^{1/4} U P^{-1/4}$$

$$= U - \tfrac{4}{3} U$$

$$= -\tfrac{1}{3} U$$

e)

Problem 5 (10 Points):

A crystal lattice consists of N atoms. Each atom is in a quantum state in which the total orbital angular momentum is zero and the total spin angular momentum is S$= \frac{1}{2}$. The crystal is in an external magnetic field $\mathbf{B}_0 = \mu_0 \mathbf{H}$ of magnetic field intensity $\mathbf{H}$, where μ_0 is the permeability of free space. Choosing the z-axis to lie along the field, we can specify a microstate in terms of the site indices σ_i for each lattice site j, which are defined as $\sigma_j = \pm 1$ if $(\mathrm{M}_s)_j = \mp\frac{1}{2}$, respectively. In the Ising model, the energy E_p for a microstate ψ_p of a one-dimensional crystal in this field is,

$$E_p = -J \sum_{(i,j)_{nn}}^{N} \sigma_i \sigma_j - B_0 \sum_{j=1}^{N} \sigma_j .$$

where $\mathrm{J} > 0$ is a constant, and the subscript $(i,j)_{nn}$ means to sum *once* over each nearest neighbor pair of sites. Now, define

$$N_+ = \text{number of atoms with } \sigma_j = 1$$
$$N_- = \text{number of atoms with } \sigma_j = -1$$
$$N_{++} = \text{number of nearest} - \text{neighbor pairs } (i,j) \text{ with } \sigma_i = 1 \text{ and } \sigma_j = 1$$
$$N_{+-} = \text{number of nearest} - \text{neighbor pairs } (i,j) \text{ with } \sigma_i = 1 \text{ and } \sigma_j = -1$$

In terms of these quantities, the microstate energy for E_p can be written

$$E_p = -4JN_{++} + 2(fJ - B_0)N_+ - \frac{1}{2}(fJ - 2B_0)N,$$

where f is defined so that the number of nearest neighbor pairs with at least one $\sigma_i = 1$ is $f\,\mathrm{N}_+ = 2\,\mathrm{N}_{++} + \mathrm{N}_{+-}$.

a. Write down expressions for the Helmholtz potential F(T,B_0,N)in terms of the canonical partition function Z(T,B_0,NJ), and for the partition function in terms of the microstate energies E_p. Do not try to evaluate or simplify your expression. **(3 Points)**

b. Let m_0 denote the difference between the fraction of atoms with $M_s = \frac{1}{2}$ and the fraction with $M_s = -\frac{1}{2}$; $m_0 = (N_+ - N_-)/N$. Derive the following approximate implicit equation for m_0 in the limit of zero field strength ($H \to 0$):

$$B_0 + fJm_0 = k_B T \tanh^{-1}(m_0)$$

(2 Points)

c. From the expression in b., derive an expression for the critical temperature T_c for spontaneous magnetization. Express your answer in terms of f, J, and Boltzmann's constant. **(2 Points)**

d. Derive the *value* of the critical exponent β (the degree of the coexistence curve) that describes how the order parameter $M_0(T)$ behaves as the temperature T approaches the critical temperature T_c from below:

$$M_0 \sim (\frac{T - T_c)}{T_c}) \text{ for } T \leq T_c$$

(3 Points)

Problem 6 (10 Points):

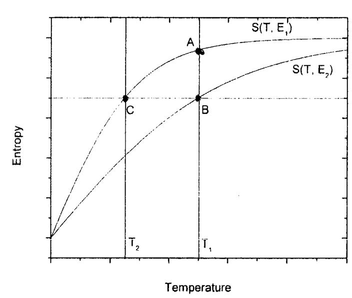
A sample consists of N independent electric dipoles. Each dipole has two possible quantum states with energies $\pm\mu E$ where E is the magnitude of an externally applied electric field. The lower energy state has dipole moment μ and the higher energy state has dipole moment $-\mu$.

a. Find the total electric dipole moment of the sample in an electric field E at temperature T. **(2 Points)**

b. What is the entropy of the sample? **(2 Points)**

c. Without using your result in b. explain physically what the entropy should be in the limits of $E \to 0$ and $E \to \infty$. **(2 Points)**

Entropy versus temperature curves for two values of electric field are shown below. Imagine that the sample is initially at state A, with temperature T_1 and field E_1.



d. How much heat must be extracted from the sample to move it from state A to state B, maintaining its temperature at T_1 while the field is raised from E_1 to E_2? **(2 Points)**

e. Once the sample is in state B, it is thermally isolated and the field is slowly reduced from E_2 to E_1, bringing the system from state B to state C. What is the temperature of the sample once it reaches state C in terms of the other variables given in the problem? **(2 Points)**

6.

$$\epsilon_i = \pm \mu_0 E$$

a) $$Z_1 = \sum_i e^{-\beta \epsilon_i} = e^{-\beta \mu E} + e^{+\beta \mu E} = 2\cosh(\beta \mu E)$$

$$Z = Z_1^N = (2\cosh(\beta \mu E))^N$$

$$\langle \mu \rangle = -\frac{\partial F}{\partial E} = KT \frac{\partial \ln Z}{\partial E} = 2NKT \frac{\sinh(\beta \mu_0 E)}{\cosh(\beta \mu_0 E)} \mu_0 \beta$$

$$\langle \mu \rangle = 2N\mu_0 \tanh(\mu_0 \beta E)$$

b) $$F = E - TS$$

$$\Rightarrow S = \frac{1}{T}(E - F) = K\beta\left[-\frac{\partial}{\partial \beta}\ln Z + \frac{1}{\beta}\ln Z\right]$$

$$= K\left[-\beta \frac{\partial \ln Z}{\partial \beta} + \ln Z\right]$$

$$= -2NK\beta \tanh(\beta \mu_0 E)\mu_0 E + 2NK \ln(\cosh(\beta \mu_0 E))$$

$$= 2NK\left[\ln(\cosh(\beta \mu_0 E)) - \beta \mu_0 E \tanh(\beta \mu_0 E)\right]$$

c) As $E \longrightarrow \infty$ the entropy would be minimum

as the system would have maximum order

which would correspond to a single microstate

As, $E \rightarrow 0$, entropy minimum, as the dipoles

would be randomly oriented

d) $$\Delta S = \frac{Q}{T_1} \Rightarrow Q = T_1\left(S(E_1) - S(E_2)\right)$$

e) $$\Delta S = 0$$

$$S(E_1, T_1) = S(E_2, T_2)$$

Solve for T_2