# Classical Mechanics and Statistical/Thermodynamics

August 2011

# Possibly Useful Information

Handy Integrals:

$$\int_0^\infty x^n e^{-\alpha x}\,dx = \frac{n!}{\alpha^{n+1}}$$

$$\int_0^\infty e^{-\alpha x^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$$

$$\int_0^\infty x\; e^{-\alpha x^2}\,dx = \frac{1}{2\alpha}$$

$$\int_0^\infty x^2\; e^{-\alpha x^2}\,dx = \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}}$$

$$\int_{-\infty}^\infty e^{i\,a\,x - b\,x^2}\,dx = \sqrt{\frac{\pi}{b}}e^{-a^2/4b}$$

Geometric Series:

$$\sum_{n=0}^\infty x^n = \frac{1}{1-x} \qquad \text{for} \quad |x| < 1$$

Stirling's approximation:

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

Riemann and related functions:

$$\sum_{n=1}^\infty \frac{1}{n^p} \equiv \zeta(p)$$

$$\sum_{n=1}^\infty \frac{z^p}{n^p} \equiv g_p(z) \qquad \sum_{n=1}^\infty (-1)^p\,\frac{z^p}{n^p} \equiv f_p(z)$$

$$g_p(1) = \zeta(p) \qquad f_p(1) = \zeta(-p)$$

Moments of Inertia:

$$I_{\text{hoop}} = MR^2$$
$$I_{\text{disk}} = \tfrac{1}{2}MR^2$$
$$I_{\text{sphericalshell}} = \tfrac{2}{3}MR^2$$
$$I_{\text{ball}} = \tfrac{2}{5}MR^2$$

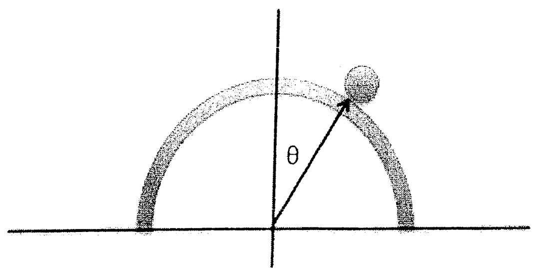
# Classical Mechanics

1. A solid uniform marble with mass $m$ and radius $r$ starts from rest on top of a hemisphere with radius $R$. It will start to roll to the right, and eventually fly off the hemisphere.

   (a) Assume that the marble rolls without slipping at all times. Calculate $\theta_1$, the angle with respect to the vertical at which the marble loses contact with the hemisphere. (3pts).

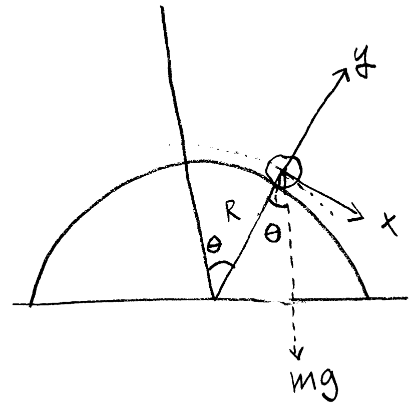
   (b) Where will the marble hit the ground, as measured from the center of the hemisphere? You may use the variable $\theta_1$ in your answer. (If you do not solve part (a), you can still attempt this problem by writing your answer in terms of this variable.) (3pts).

   (c) Now assume that the force of friction between marble and the hemisphere is $\mu N$, where $N$ is the normal force between the marble and the hemisphere. Calculate the angle $\theta_2$ at which the marble will no longer roll without slipping. (4pts).



1.

$\Sigma F_y = -ma_c$ $\qquad r' = R + r$




$\Rightarrow N - mg\cos\theta = -\dfrac{mv^2}{r'}$

When the marbel loses contact

$N = 0$

$\Rightarrow mg\cos\theta_1 = \dfrac{mv^2}{r'}$

$\Rightarrow gr'\cos\theta_1 = v^2$

Using conservation of energy

$mgr' = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + mgr'\cos\theta_1$

$= \frac{1}{2}mv^2 + \frac{1}{5}mr^2\frac{v^2}{r^2} + mgr'\cos\theta_1$

$= \frac{1}{2}mv^2 + \frac{1}{5}mv^2 + mgr'\cos\theta_1$

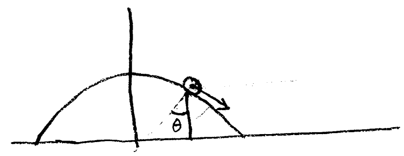
$gr' = \frac{7}{10}v^2 + gr'\cos\theta_1$

$\Rightarrow v^2 = \frac{10}{7}gr'(1-\cos\theta_1)$

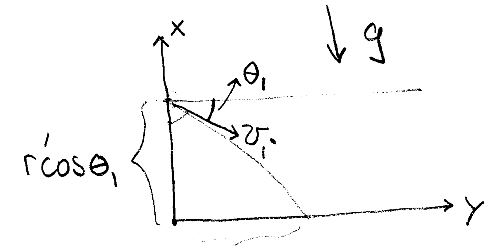
Thus, $\cos\theta_1 = \frac{10}{7}(1-\cos\theta_1)$ $\Rightarrow \frac{17}{7}\cos\theta_1 = \frac{10}{7}$

$\Rightarrow \theta_1 = \cos^{-1}\left(\frac{10}{17}\right)$

(b) $\Rightarrow v_i^2 = \frac{10}{7} g r' (1 - \cos\theta_1)$




$\cos\theta_1 = \frac{10}{17}$ $\quad \sin^2\theta_1 = \frac{49}{100}$




$v_i^2 = \frac{10}{7} g r' \frac{7}{17} = \frac{10}{17} g r'$

$\Rightarrow v_i = \sqrt{\frac{10}{17} g r'}$

$x_f = x_i + v_{i,x} t$

$\Rightarrow \Delta x = v_{i,x} t$

$\Rightarrow \Delta x = v_i \cos\theta_1 \frac{2 v_i \sin\theta_1}{g}$

$\times \left[ 1 + \sqrt{1 + \frac{34}{10} \frac{\cos\theta_1}{\sin^2\theta_1} r'} \right]$

$= \frac{v_i^2 \sin 2\theta_1}{g} \left[ 1 + \frac{34}{10} \frac{\cos\theta_1}{\sin^2\theta_1} r' \right]^{1/2}$

$\frac{\cos\theta_1}{\sin^2\theta_1} = \frac{10}{17} \times \frac{100}{49}$

$= \frac{10\sqrt{2r'}}{7} \frac{v_i^2 \sin 2\theta_1}{g}$

$y_f = y_i + v_{i,y} t - \frac{1}{2} g t^2$

$-r' \cos\theta_1 = -v_i \sin\theta_1 t - \frac{1}{2} g t^2$

$\Rightarrow t = \frac{-v_i \sin\theta_1 \pm \sqrt{v_i^2 \sin^2\theta_1 + 2 g r' \cos\theta_1}}{\frac{1}{2} g}$

$\Rightarrow t = \frac{-2 v_i \sin\theta_1 \pm 2\sqrt{v_i^2 \sin^2\theta_1 - 2 g r' \cos\theta}}{g}$

$\Rightarrow t = \frac{-2 v_i \sin\theta_1 \pm 2 v_i \sin\theta_1 \sqrt{1 + \frac{34}{10} \frac{\cos\theta_1 r'}{\sin^2\theta_1}}}{g}$

(c) when it is only rolling

$$f_s \leq \mu_s N$$

* when it starts slipping $f_s = \mu_s N$

$$\Sigma F_y = -ma_c$$

$$\Rightarrow \quad N - mg\cos\theta = -\frac{mv^2}{r'}$$

* Strauss's suggestion

$$\tau = F_f r = \mu N r = I\dot{\omega} = I\alpha$$

$$V = \omega r$$

$$\Rightarrow \quad \frac{\dot{V}}{r} = \dot{\omega}$$

2. Consider a point particle of mass $m$ moving under the influence of a central force:

$$\vec{F}(\vec{r}) = -\frac{k}{r^n}\,\hat{r}$$

where $n$ is an integer greater than one ($n = 2, 3, \ldots$), the variable $r$ is the distance from the origin of the force ($r \equiv |\vec{r}|$) and $\hat{r}$ is a unit vector in the radial direction. In this problem, we will examine when circular orbits are stable for such a central force.

   (a) Calculate potential energy of this force. Choose the zero of the potential to be at infinity ($r = \infty$). (1pt)

   (b) Show that the angular momentum about the origin, $L$, is conserved. (You may use the Newtonian, Lagrangian, or Hamiltonian formulations of the problem). (2pts)

   (c) Write an expression for the total energy of the particle $E$ as a function of $r$, $dr/dt$, $L$, $k$, and $n$. (1pt)

   (d) Assume the particle is moving in a circular orbit about the origin, so that $dr/dt = 0$. Calculate the radius of the orbit and the velocity of the particle as a function of the above variables. (3pts)

   (e) When is this circular orbit stable? (Hint: look at $dE/dr$ and $d^2E/dr^2$.) (3pts)

2.

$$\bar{F}(\bar{r}) = -\frac{k}{r^n}\hat{r} \qquad n > 1$$

a) $$\bar{F} = -\frac{\partial V(r)}{\partial r}\hat{r}$$

$$\Rightarrow \quad V(r) = -\int F\,dr = \int_{\infty}^{r} \frac{K}{r^n}\,dr$$

$$= \frac{K}{-n+1} r^{-n+1}\Big|_{\infty}^{r}$$

$$V(r) = \frac{K}{-n+1} r^{-n+1}$$

b) $$\mathcal{L} = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{2}mr^2\sin^2\theta\,\dot{\phi}^2 - \frac{K}{-n+1}r^{-n+1}$$

$\phi$ is cyclic

$\dot{p}_\phi = 0$ $\qquad$ $p_\phi$ is const.

c) Choose $\theta = \pi/2$

$$H = \frac{p_r^2}{2m} + \frac{\overset{\text{const}}{p_\phi^2}}{2mr^2} + \frac{K}{-n+1}r^{-n+1}$$

no time dependence

$$E = \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + \frac{p_\phi^2}{2mr^2} + \frac{Kr^{-n+1}}{-n+1} \qquad p_\phi = mr^2\dot{\phi}$$

d) $$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m}$$

$$\dot{p}_r = -\frac{\partial H}{\partial r} = -\frac{p_\phi^2}{mr^3} + kr^{-n}$$

$$\rightsquigarrow m\ddot{r} = -\frac{p_\phi^2}{mr^3} + kr^{-n}$$

$$\Rightarrow \frac{k}{r^n} = \frac{p_\phi^2}{mr^3}$$

$$\Rightarrow r^{n-3} = \frac{mk}{p_\phi^2}$$

$$\Rightarrow r_0 = \left(\frac{mk}{p_\phi^2}\right)^{\frac{1}{n-3}}$$

$$\Rightarrow m\frac{d\dot{r}}{dr}\dot{r} = -\frac{p_\phi^2}{mr^3} + kr^{-n}$$

$$\Rightarrow \frac{m\dot{r}^2}{2} = +\frac{p_\phi^2}{2mr^2} + \frac{k}{-n+1} r^{-n+1}$$

$$\Rightarrow \dot{r}^2 = \frac{p_\phi^2}{m^2r^2} + \frac{2kr^{-n+1}}{(-n+1)m}$$

$$\Rightarrow \dot{r} = \sqrt{\frac{p_\phi^2}{m^2r^2} + \frac{2kr^{-n+1}}{(-n+1)m}}$$

but $\dot{r} = 0$

& $$\dot{\phi} = \frac{p_\phi}{mr^2}$$

d) $$E = \frac{p_\phi^2}{2mr^2} + \frac{k r^{-n+1}}{-n+1}$$

$$\frac{dE}{dr} = -\frac{p_\phi^2}{mr^3} + k r^{-n}$$

$$\frac{d^2E}{dr^2} = \left[ + \frac{3p_\phi^2}{mr^4} - n k r^{-n-1} \right]_{r=r_0}$$

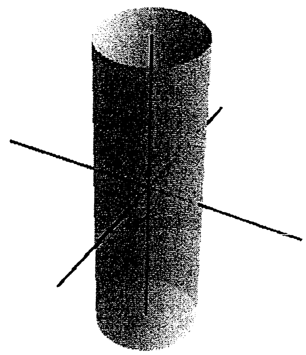
$$= \frac{3p_\phi^2}{m} \left(\frac{mk}{p_\phi^2}\right)^{-\frac{4}{n-3}} - nk \left(\frac{mk}{p_\phi^2}\right)^{\frac{-(n+1)}{n-3}} > 0$$

$$3\frac{p_\phi^2}{m} > + nk \left(\frac{mk}{p_\phi^2}\right)^{\overbrace{\frac{-n-1+4}{n-3}}^{=-1}}$$

$$\Rightarrow \quad 3\frac{p_\phi^2}{m} > nk \frac{p_\phi^2}{mk}$$

$$\Rightarrow \quad n < 3$$

3. A particle of mass $m$ is constrained to move on an infinitely long cylinder of radius $a$. The center of the cylinder is oriented along the z-axis, as shown. An attractive central potential, $U(r) = U(\sqrt{a^2 + z^2})$, is located at the origin, where $r$ is the radius is spherical coordinates.



(a) Write down the Lagrangian for the problem. (1pt)

(b) From the Lagrangian, explicitly derive the Hamiltonian for the particle. (2pts)

(c) Is angular momentum about the z-axis conserved? Prove your answer. (2pts)

(d) Under what conditions is motion in the z-direction bounded? (2pts)

(e) Assume that the potential is $U(r) = \frac{1}{2}\alpha r^2$. Solve the equations of motion, and reduce the problem to quadrature. (3pts)

3.

a) $L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(x,y,z)$

$$x = a\cos\theta \Rightarrow \dot{x} = -a\sin\theta\,\dot{\theta}$$

$$y = a\sin\theta \Rightarrow \dot{y} = a\cos\theta\,\dot{\theta}$$

$$L = \frac{1}{2}m\left(a^2\sin^2\theta\,\dot{\theta}^2 + a^2\cos^2\theta\,\dot{\theta}^2 + \dot{z}^2\right) - \underbrace{U\left(\sqrt{a^2+z^2}\right)}_{\shortparallel}$$

$$= \frac{1}{2}ma^2\dot{\theta}^2 + \frac{1}{2}m\dot{z}^2 - U\left(\sqrt{a^2+z^2}\right)$$

b) $p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ma^2\dot{\theta} \Rightarrow \dot{\theta} = \frac{p_\theta}{ma^2}$

$$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} \Rightarrow \dot{z} = \frac{p_z}{m}$$

$$H = \sum p_i \dot{q}_i - L$$

$$= ma^2\dot{\theta}^2 + m\dot{z}^2 - \frac{1}{2}ma^2\dot{\theta}^2 - \frac{1}{2}m\dot{z}^2 + U\left(\sqrt{a^2+z^2}\right)$$

$$= \frac{1}{2}ma^2\dot{\theta}^2 + \frac{1}{2}m\dot{z}^2 + U\left(\sqrt{a^2+z^2}\right)$$

$$= \frac{p_\theta^2}{2ma^2} + \frac{p_z^2}{2m} + U\left(\sqrt{a^2+z^2}\right)$$

c) $V(r) = V(\sqrt{a^2+z^2})$ the potential is a func of $z$

$a$ is a const

Thus,

$$\frac{\partial L}{\partial \theta} = \frac{\partial U(z)}{\partial \theta} = 0$$

$$\therefore \quad \frac{dP_\theta}{dt} = 0$$

or Hamilton's Eqn of motion

$$\dot{P}_\theta = -\frac{\partial H}{\partial \theta} = -\frac{\partial U(z)}{\partial \theta} = 0$$

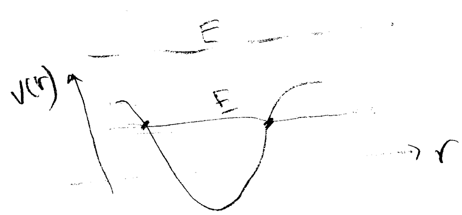
Hence, the angular momentum is conserved about $z$-axis

d) * the potential is acting towards the center

& at some $z$ is will be infinite!

$$P_z \overset{?}{=} 0$$

*



e) $$V(r) = \frac{1}{2}\alpha r^2 = \frac{1}{2}\alpha(a^2+z^2)$$

$$H = \frac{p_\theta^2}{2ma^2} + \frac{p_z^2}{2m} + \frac{1}{2}\alpha(a^2+z^2)$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \quad \& \quad \dot{q}_i = \frac{\partial H}{\partial p_i}$$

$$\dot{p}_z = -\frac{\partial H}{\partial z} = -z$$

$$\dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m}$$

Thus, $$\ddot{p}_z = -\dot{z} = -\frac{p_z}{m}$$

$$\Rightarrow \ddot{p}_z + \frac{p_z}{m} = 0$$

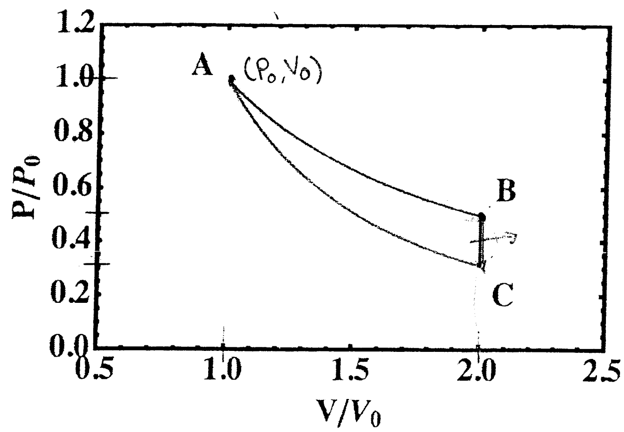
$$\ddot{z} = \frac{\dot{p}_z}{m}$$

$$\Rightarrow \ddot{z} + \frac{z}{m} = 0$$

* quadrature $q(t) = \int f(\,)$ $\Rightarrow$ min required

# Statistical Mechanics

4. Consider an ideal monatomic gas used as the working fluid in a thermodynamic cycle. The number of particles is $n_0$. It follows a cycle consisting of one adiabat, one isochore and one isotherm, as shown below.




(a) Calculate the pressure, temperature, and volume at each corner of the cycle, A, B, and C, expressing your answer in terms of $P_0$, $V_0$, $n_0$ and perhaps $R$, the ideal gas constant. Note that point A the pressure is $P_0$ and the volume is $V_0$. (3pts)

(b) Calculate the work done on the system, the heat into the system and the change in the internal energy of the system for each process step. (4.5pts)

(c) What direction around thecycle must the system follow to be used as a functional heat engine? (1/2pt)

(d) What is the efficiency of the cycle, run as an engine? (1pt)

(e) What is the efficiency of an ideal Carnot engine run between reservoirs B and C? (1pt)

* adiabat is steeper than isotherm

4.

- $A \to B \Rightarrow$ isotherm
- $B \to C \Rightarrow$ isochore
- $C \to A \Rightarrow$ adiabat

a) • <u>A→B</u>

$$\frac{P_A V_A}{T_A} = \frac{P_B V_B}{T_B}$$

$$\frac{P_B}{P_0} = \frac{1}{2} \Rightarrow P_B = 0.5P_0$$

$$\frac{V_B}{V_0} = 2 \Rightarrow V_B = 2V_0$$

$$\Rightarrow \frac{P_0 V_0}{T_A} = \frac{P_0 V_0}{T_B}$$

$$A \equiv (P, V, T) = \left(P_0, V_0, \frac{P_0 V_0}{n_0 K}\right)$$

$$\Rightarrow T_A = T_B$$

$$\text{but } T_A = \frac{P_A V_A}{n_0 K} = \frac{P_0 V_0}{n_0 K}$$

$$B \equiv (P, V, T) = \left(\tfrac{1}{2}P_0, 2V_0, \frac{P_0 V_0}{n_0 K}\right)$$

$n$ → #moles

$$nR = n_0 K$$

$$\Rightarrow K = \frac{n}{n_0} R = \frac{n_0/N_A}{n_0} R = \frac{R}{N_A}$$

• <u>B→C</u>

$$\frac{P_B}{T_B} = \frac{P_C}{T_C}$$

$$P_C = 0.3 P_0$$

$$\Rightarrow \frac{0.5P_0}{\frac{P_0 V_0}{n_0 K}} = \frac{0.3P_0}{T_C}$$

$$\Rightarrow T_C = \frac{0.3 P_0 V_0}{0.5 n_0 K} = \frac{3P_0 V_0}{5 n_0 K}$$

$$C \equiv (P, V, T) = \left(0.3P_0, 2V_0, \frac{3P_0 V_0}{5 n_0 K}\right)$$

• <u>A→C</u>

$$P_A V_A^{\gamma} = P_C V_C^{\gamma}$$

$$P_0 V_0^{\gamma} = 2^{\gamma} P_C V_0^{\gamma}$$

$$\Rightarrow P_C = 2^{-\gamma} P_0 = 2^{-3/2} P_0$$

b) $\underline{A \to B}$

$$W_1 = -\int P\,dV = -\int_{V_i}^{V_f} \frac{nRT}{V}\,dV = nRT \ln\left(\frac{V_i}{V_f}\right)$$

$$= -nRT \ln 2 = -P_0 V_0 \ln 2$$

$$\Delta E = \frac{3}{2} n_0 K \Delta T = 0; \quad Q = \Delta E - W_1 = P_0 V_0 \ln 2$$

$\underline{B \to C}$

$$W_2 = 0 \qquad Q_2 = C_V \Delta T = C_V(T_C - T_B) = -0.4\, C_V \frac{P_0 V_0}{n_0 K}$$

$$\Delta E = -0.4\, C_V \frac{P_0 V_0}{n_0 K} \text{ or } \underline{\Delta E = \frac{3}{2} n_0 K \Delta T} = -0.6 P_0 V_0$$

$\underline{C \to A}$

$$W_3 = -\int_{V_C}^{V_A} P\,dV = +\Delta E_{int} = \frac{3}{2} n_0 K (T_A - T_C)$$

$$= \frac{3}{2}(P_0 V_0 - 0.6 P_0 V_0)$$

$$= \frac{3}{2} 0.4 P_0 V_0$$

$$= \underline{\frac{3}{5} P_0 V_0}$$

$$Q_3 = 0$$

$$\Delta E = \frac{3}{5} P_0 V_0$$

c) Clockwise $\Rightarrow$ $W_{\text{net on the gas}} < 0$ $\therefore$ $W_{\text{net by the gas}} > 0$

Hence, clockwise

d) $Q_{in} = P_0 V_0 \ln 2$

$Q_{out} = -0.6 P_0 V_0$

$$\eta = \frac{|Q_{in}| - |Q_{out}|}{Q_{in}} \approx \frac{0.1 P_0 V_0}{0.7 P_0 V_0} \approx 14\%$$

e) $$\eta = 1 - \frac{T_C}{T_B} = 1 - \frac{3P_0V_0/5n_0K}{\frac{P_0V_0}{n_0K}} = 40\%$$

5. Consider the quantum mechanical linear rotator. It has energy levels

$$E_J = \frac{\hbar^2}{2I} J(J+1)$$

where $I$ is the moment of inertia and $J$ is the angular momentum quantum number, $J = 0, 1, 2, \ldots$. Each energy level is $(2J+1)$-fold degenerate.

(a) In the low temperature limit ($\hbar^2/2I \gg kT$) determine approximate expressions for:

i. The rotation partition function. (2pts)

ii. The internal energy. (1pt)

iii. The specific heat. (1pt)

(b) In the high temperature limit ($\hbar^2/2I \ll kT$) determine approximate expressions for:

i. The rotation partition function. (2pt)

ii. The internal energy. (1pt)

iii. The specific heat. (1pt)

(c) How do the quantum results compare with the equipartition theorem for a classical rotator with two transverse degrees of freedom? (2pts)

5.

The energy levels of the rotator is

$$E_J = \frac{\hbar^2}{2I} J(J+1)$$

So, the partition func is

$$Z = \sum_J (2J+1)\, e^{-\beta \underbrace{\frac{\hbar^2}{2I}}_{\alpha} J(J+1)}$$

a) In low temp limit

$(\hbar^2/2I)\beta >> 1$ so, $e^{-\alpha\beta J(J+1)} \to$ for higher J values the contribution to the sum is negligible

i. Thus,

$$Z \approx e^0 + 3e^{-2\alpha\beta} + 5e^{-6\alpha\beta} + 7e^{-12\alpha\beta}$$

$$= 1 + 3e^{-2\alpha\beta} + 5e^{-6\alpha\beta} + 7e^{-12\alpha\beta} + \cdots$$

ii. $E = -\frac{\partial \ln Z}{\partial \beta}$

$$= -\frac{1}{Z}\left[-6\alpha e^{-2\alpha\beta} - 30\alpha e^{-6\alpha\beta} - 84\alpha e^{-12\alpha\beta}\right]$$

$$= \frac{\alpha}{Z}\left(6e^{-2\alpha\beta} + 30e^{-6\alpha\beta} + 84e^{-12\alpha\beta}\right)$$

iii. $C_V = \frac{\partial E}{\partial T} = \left(-\frac{1}{KT^2}\right)\frac{\partial E}{\partial \beta} = -K^2\beta^2 \frac{\partial E}{\partial \beta}$

$$= -K^2\beta^2 \frac{\alpha}{Z}\left(-12\, e^{-2\alpha\beta} - 180\, e^{-6\alpha\beta} - \cdots\right)$$

$$= \frac{\alpha}{Z} K^2\beta^2\left(12\, e^{-2\alpha\beta} + 180\, e^{-6\alpha\beta} + \cdots\right)$$

⇒ Simplfy

$$E = \frac{\hbar^2}{2I} \frac{6e^{-2\alpha\beta} + 30\, e^{-6\alpha\beta}}{(1 + 3e^{-2\alpha\beta} + 5e^{-6\alpha\beta} + \cdots)}$$

b) In high temp limit

$(\hbar^2/2I)\beta \ll 1$, the spacing between energy levels are too small compared to the total energy, So we can consider the energy levels are essentially continuous

$J^2 + J = u$

$(2J+1)dJ = du$

$J = 0, \; u = 0$

$J = \infty, \; u = \infty$

So,

$$Z = \int_0^\infty (2J+1)\, e^{-\beta\alpha J(J+1)}\, dJ$$

$$= \int_0^\infty du\, e^{-\beta\alpha u}$$

$$= \frac{1}{\alpha\beta}$$

$$= \frac{2I}{\hbar^2}\frac{1}{\beta}$$

ii. $E = -\frac{\partial \ln Z}{\partial \beta}$

$$= -\frac{\partial}{\partial\beta}\ln\left(\frac{2I}{\hbar^2}\frac{1}{\beta}\right)$$

$$= -\frac{\hbar^2\beta}{2I}\left(-\frac{1}{\beta^2}\right)\frac{2I}{\hbar^2}$$

$$= KT$$

III. $C_V = \frac{\partial E}{\partial T} = K$

c) For a classical rotator with two degrees of freedom from equipartition theorem we expect,

$$E = 2\frac{1}{2}KT = KT$$

$$C_V = K$$

For a quantum rotator at high temp limit this is exactly what we got, since our rotator has two degrees of freedom (i.e angular momentum & spin)

6. Consider the "bogon," a spin 5/2 fermion with the charge of an electron but with a dispersion relationship

$$E = cp^3.$$

where $p \equiv |\vec{p}|$ Assume that your bogons are confined in a three dimensional sample and are non-ineracting.

(a) Working in the grand canonical ensemble, determine the density, $\rho = \langle N \rangle / V$, as a function of the chemical potential, $\mu$ (or the fugacity, $z \equiv e^{\beta\mu}$), $T$, and $V$. (3pts)

(b) What is the bogonic Fermi energy ($\mu$ at $T = 0$) as a function of their density? (3pts) (*Hint:* This should not involve any complicated integrals).

(c) Derive a series expansion in $z$ for the grand canonical free entropy, $\Xi = \frac{PV}{kT} = \log \mathcal{Z}$, where $\mathcal{Z}$ is the grand canonical partition function. (4pts)

6.

the energy of the spin 5/2 fermion is

$$\varepsilon = cp^3 \Rightarrow E = \sum_i n_i \epsilon_i$$

the partion function can be written as,

$$Q(N,T,V) = \sum_{N=0}^{\infty} \sum_{\{\sigma\}} e^{\beta \mu N} e^{-\beta E_\sigma}$$

$$= \sum_{N=0}^{\infty} \sum_{\{n_{i,\sigma}\}} e^{\beta\mu \sum_{i,\sigma} n_i} e^{-\beta \sum_{i,\sigma} n_i \epsilon_i}$$

$$= \sum_{n_i=0}^{\infty} e^{\sum_{i=0, \sigma}^{\infty} n_i \beta(\mu - \epsilon_i)}$$

$$= \sum_{n_i=0}^{\infty} \prod_{i,\sigma} e^{-(\epsilon_i - \mu) n_i}$$

$$= \prod_i \left( \sum_{n_i=0}^{\infty} e^{-(\epsilon_i - \mu) n_i} \right)^{(2s+1)}$$

but for fermions $n_i = 0$ or $1$

$$Q(N,T,V) = \prod_i \left( 1 + e^{-\beta(\epsilon_i - \mu)} \right)^{(2s+1)}$$

₩

The Grand potential is

$$\mathcal{G} = -KT \ln Q$$

$$= -6KT \ln \left\{ \prod_i \left( 1 + e^{-\beta(\epsilon_i - \mu)} \right) \right\}$$

$$= -6KT \sum_{i=0}^{\infty} \ln \left( 1 + e^{-\beta(\epsilon_i - \mu)} \right)$$

$$\mathcal{G} = F - \mu N$$

$$\mathcal{G} = E - TS - \mu N$$

$$\Rightarrow d\mathcal{G} = + T\,dS - P\,dV + \mu\,dN - T\,dS - S\,dT - \mu\,dN - N\,d\mu$$

$$d\mathcal{G} = -P\,dV - S\,dT - N\,d\mu$$

$$\langle N \rangle = - \left. \frac{\partial \mathcal{G}}{\partial \mu} \right|_{V,T}$$

as $s = \frac{5}{2}$

$$= KT \sum_i \frac{6 \beta e^{-\beta(\epsilon_i - \mu)}}{1 + e^{-\beta(\epsilon_i - \mu)}}$$

$$= 6 \sum_i \frac{1}{e^{\beta(\epsilon_i - \mu)} + 1}$$

$$\langle N \rangle = \sum_i \frac{6}{e^{\beta(cp^3 - \mu)} + 1}$$

↗ sum of all the quantum states

$$\approx 6 \frac{V}{(2\pi\hbar)^3} \int_0^\infty d^3p \frac{1}{e^{\beta(cp^3-\mu)} + 1}$$

$$\frac{\langle N \rangle}{V} = 6 \frac{4\pi}{(2\pi\hbar)^3} \int_0^\infty dp\, p^2 \underbrace{\frac{1}{e^{\beta(cp^3-\mu)} + 1}}_{f(E)}$$

b)

at $T = 0$, $\mu = E_F$

$$f(E) = \theta(E_F - E)$$

Now, $E = cp^3$

$$\Rightarrow dE = 3cp^2 dp$$

$$\rho = 6 \frac{4\pi}{(2\pi\hbar)^3} \frac{1}{3c} \int_0^\infty dE\, \theta(E_F - E)$$

$$\rho = \frac{8\pi}{(2\pi\hbar)^3 c} E_F \qquad \Rightarrow E_F = \frac{(2\pi\hbar)^3 c}{8\pi} \rho$$

c) Grand canonical free entropy

$$S = -\frac{\partial \mathcal{G}}{\partial T}\Big|_{V,\mu}$$

$\Xi \overset{?}{=} \frac{PV}{KT} \overset{?}{=} \log Q$

↓

this is not entropy but

* Grand canonical potential

$$S = +\frac{\partial}{\partial T}\{KT \ln Q\}$$

$$= \ln Q + KT \frac{\partial \ln Q}{\partial T}$$

$$\Xi = \frac{PV}{KT} = \ln Q$$

$$\Rightarrow \frac{PV}{KT} = \sum_{i=0}^{\infty} \ln\left(1 + e^{-\beta(\epsilon_i - \mu)}\right)$$

$$= \sum_{i=0}^{\infty} \ln\left(1 + e^{-\beta\epsilon_i} e^{\beta\mu}\right)$$

$$= \sum_{i=0}^{\infty} \ln\left(1 + z e^{-\beta\epsilon_i}\right)$$

In view that for large volume the single particle energy spectrum is almost continous

$$\approx \frac{V\,4\pi}{(2\pi\hbar)^3}\int_0^{\infty} p^2 dp \ln(1 + z e^{-\beta\epsilon_i})$$

Now, $\epsilon = cp^3$

$\Rightarrow d\epsilon = 3cp^2 dp$

$$\Rightarrow \quad \frac{PV}{KT} \approx \frac{V}{(2\pi\hbar)^3} 4\pi \, 3c \int_0^\infty d\epsilon \, \ln\left(1 + z e^{-\beta\epsilon}\right)$$

$z$ is small in the expansion

$$= \frac{12\pi c V}{(2\pi\hbar)^3} \int_0^\infty d\epsilon \left\{ 1 + z e^{-\beta\epsilon} + \frac{z^2 e^{-2\beta\epsilon}}{2!} + \cdots \right\}$$