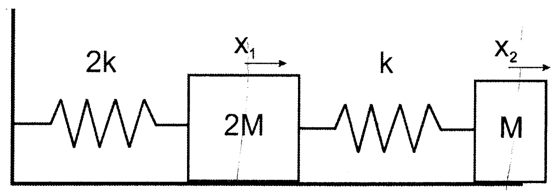
Mechanics and Statistical Mechanics Qualifying Exam
Fall 2010

Problem 1: (10 Points)

Two blocks are free to move in *one* dimension along a frictionless horizontal surface. The blocks of mass 2M and M are connected to each other and to a fixed wall by two springs with stiffness 2k and k as shown in the figure. Choose the dynamical coordinates of the system to be the position of block 1, x_1, and block 2, x_2, from their respective equilibrium positions. Consider only small oscillations so that the springs are linear. Neglect all damping.



a. Write down the equations of motion for each mass. **(2 Points)**

b. Show that the frequencies of the normal modes of the system are $\sqrt{2}\omega_0$ and and $\omega_0/\sqrt{2}$ where $\omega_0 = \sqrt{k/m}$. **(2 Points)**

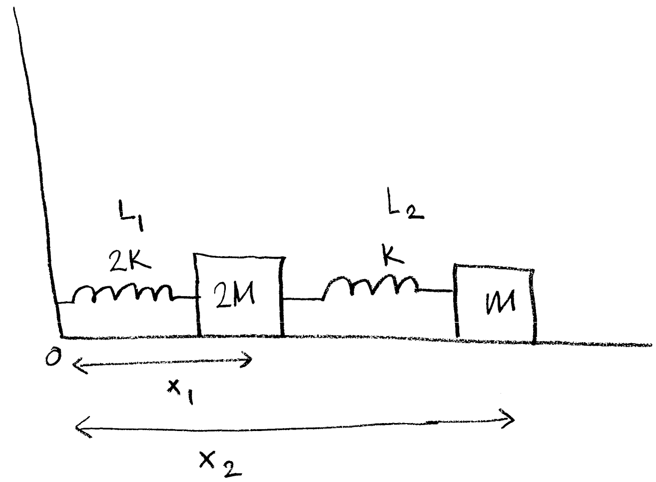
c. Find the eigenvectors that describe the normal modes and sketch them. **(3 Points)**

d. Suppose you grab mass M and push it slowly to the left by an amount A_0. When mass 2M is in equilibrium show that it is $A_0/4$ from its equilibrium position. **(1 Point)**

e. If you release the system from the starting position in (d.), what will be the displacement of the system as a function of time? Write an expression for the displacement of block 1 (mass 2M) as a function of time from its original equilibrium position. **(2 Points)**

1.

$$L = \frac{1}{2}(2m)\dot{x}_1^2 + \frac{1}{2}(m)\dot{x}_2^2 - \frac{1}{2}2K(x_1 - L_1)^2$$
$$- \frac{1}{2}K(x_2 - x_1 - L_2)^2$$



$$\eta_1 = x_1 - L_1 \quad \Rightarrow \dot{\eta}_1 = \dot{x}_1$$

$$\eta_2 = x_2 - L_1 - L_2 \quad \Rightarrow \dot{\eta}_2 = \dot{x}_2$$

$$\Rightarrow \quad x_1 = \eta_1 + L_1 \qquad \Rightarrow x_2 = \eta_2 + L_1 + L_2$$

$$\rightsquigarrow \quad x_1 - L_1 = \eta_1 \qquad \rightsquigarrow x_2 - x_1 - L_2 = \eta_2 + \cancel{L_1} + \cancel{L_2} - \eta_1 - \cancel{L_1} - \cancel{L_2}$$
$$= \eta_2 - \eta_1$$

So, For small oscillations,

$$L = \frac{1}{2}2m\dot{\eta}_1^2 + \frac{1}{2}m\dot{\eta}_2^2 - \frac{1}{2}2K\eta_1^2 - \frac{1}{2}K(\eta_2 - \eta_1)^2$$

$$= \frac{1}{2}(2m)\dot{\eta}_1^2 + \frac{1}{2}m\dot{\eta}_2^2 - \frac{1}{2}(3K)\eta_1^2 - \frac{1}{2}K\eta_2^2 + \frac{1}{2}(2K)\eta_1\eta_2$$

$$\tilde{T} = \begin{pmatrix} 2m & 0 \\ 0 & m \end{pmatrix} \quad \tilde{V} = \begin{pmatrix} 3K & -K \\ -K & K \end{pmatrix}$$

a) Eqn of motion for each mass

$$\rightsquigarrow 2m\ddot{\eta}_1 + 3K\eta_1 - K\eta_2 = 0$$

$$\rightsquigarrow m\ddot{\eta}_2 + K\eta_2 - K\eta_1 = 0$$

b) $$\det |\tilde{V} - \lambda \tilde{T}| = 0$$

$$\Rightarrow \begin{vmatrix} 3K - \lambda(2m) & -K \\ -K & K - \lambda m \end{vmatrix} = 0$$

$$\Rightarrow (3K - 2\lambda m)(K - \lambda m) - K^2 = 0$$

$$\Rightarrow 3K^2 - 5K\lambda m + 2\lambda^2 m^2 - K^2 = 0$$

$$\Rightarrow 2K^2 - 5K\lambda m + 2\lambda^2 m^2 = 0$$

$$\Rightarrow 2K^2 - 4K\lambda m - K\lambda m + 2\lambda^2 m^2 = 0$$

$$\Rightarrow 2K(K - 2\lambda m) - \lambda m(K - 2\lambda m) = 0$$

$$\Rightarrow (K - 2\lambda m)(2K - \lambda m) = 0$$

$$\lambda = \frac{K}{2m}, \quad \lambda = \frac{2K}{m}$$

$$\lambda_1 = \frac{K}{2m}$$

$$\lambda_2 = \frac{2K}{m} \rightarrow \text{higher freq.} \Rightarrow \text{antisymmetric mode}$$

$$\omega_1 = \sqrt{\frac{K}{2m}} = \frac{\omega_0}{\sqrt{2}} \quad ; \quad \omega_2 = \sqrt{2}\,\omega_0$$

c) For, $\lambda_1 = \frac{K}{2m}$

$$\begin{pmatrix} 3K - K & -K \\ -K & K - \frac{K}{2} \end{pmatrix} \begin{pmatrix} C_{11} \\ C_{21} \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} 2K & -K \\ -K & K/2 \end{pmatrix} \begin{pmatrix} C_{11} \\ C_{21} \end{pmatrix} = 0$$

$$\left.\begin{aligned} &\Rightarrow (2K)\,C_{11} - K\,(C_{21}) = 0 \\ &\Rightarrow -K\,C_{11} + \frac{K}{2}\,C_{21} = 0 \end{aligned}\right\} \Rightarrow C_{11} = +\frac{1}{2}C_{21}$$

$$\Rightarrow \frac{C_{11}}{C_{21}} = +\frac{1}{2} \qquad \begin{pmatrix} C_{11} \\ C_{21} \end{pmatrix} = N_1 \begin{pmatrix} 1 \\ +2 \end{pmatrix}$$

For $\lambda_2 = \frac{2K}{m}$

$$\begin{pmatrix} 3K - 4K & -K \\ -K & K - 2K \end{pmatrix}$$

$$\begin{pmatrix} -K & -K \\ -K & -K \end{pmatrix} \begin{pmatrix} C_{21} \\ C_{22} \end{pmatrix} = 0 \Rightarrow \frac{C_{21}}{C_{22}} = -1 \qquad \begin{pmatrix} C_{21} \\ C_{22} \end{pmatrix} = N_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

To calculate the normalization

$$\tilde{C}_i^T \tilde{T} \tilde{C}_i = \mathbb{1}$$

$$\underset{1\times 2}{N_1 \begin{pmatrix} 1 & +2 \end{pmatrix}} \underset{2\times 2}{\begin{pmatrix} 2m & 0 \\ 0 & m \end{pmatrix}} \underset{2\times 1}{N_1 \begin{pmatrix} 1 \\ +2 \end{pmatrix}} = \mathbb{1}$$

$$\Rightarrow N_1^2 \begin{pmatrix} 1 & +2 \end{pmatrix} \begin{pmatrix} 2m \\ +2m \end{pmatrix} = \mathbb{1}$$

$$\Rightarrow N_1^2 (2m + 4m) = 1$$

$$\Rightarrow N_1 = \frac{1}{\sqrt{6m}}$$

Similarly,

$$N_2^2 \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 2m & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \mathbb{1}$$

$$N_2^2 \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 2m \\ -m \end{pmatrix} = 1 \quad \Rightarrow \quad N_2 = \frac{1}{\sqrt{3m}}$$

$$C = \begin{pmatrix} \frac{1}{\sqrt{6m}} & \frac{1}{\sqrt{3m}} \\ +\frac{2}{\sqrt{6m}} & -\frac{1}{\sqrt{3m}} \end{pmatrix}$$

c) The general sol^n is then

$$\eta_1(t) = \begin{pmatrix} C_{11} \\ C_{21} \end{pmatrix}(A_1 \cos\omega_1 t + B_1 \sin\omega_1 t) \; ; \; \omega_1^2 > 0$$

$$\eta_2(t) = \begin{pmatrix} C_{12} \\ C_{22} \end{pmatrix}(A_2 \cos\omega_2 t + B_2 \sin\omega_2 t) \; ; \; \omega_2^2 > 0$$

or,

$$\begin{pmatrix} \eta_1(t) \\ \eta_2(t) \end{pmatrix} = \begin{pmatrix} C_{11} \\ C_{21} \end{pmatrix}[A_1 \cos\omega_1 t + B_1 \sin\omega_1 t] + \begin{pmatrix} C_{12} \\ C_{22} \end{pmatrix}[A_2 \cos\omega_2 t + B_2 \sin\omega_2 t]$$

Initial conidition $\eta(0) = \begin{pmatrix} 0 \\ A_0 \end{pmatrix} \Rightarrow \dot{\eta}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\tilde{C}_2^T \tilde{T} \dot{\eta}_2(0) = \omega_2 B_2 \Rightarrow B_2 = 0$$

$$\tilde{C}^T \tilde{T} \eta(0) = \begin{pmatrix} 1/\sqrt{6m} & +\frac{2}{\sqrt{6m}} \\ 1/\sqrt{3m} & -\frac{1}{\sqrt{3m}} \end{pmatrix} \underbrace{\begin{pmatrix} 2m & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} 0 \\ A_0 \end{pmatrix}}_{\begin{pmatrix} 2mA_0 \\ mA_0 \end{pmatrix}} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} \dfrac{2mA_0}{\sqrt{6m}} + \dfrac{2mA_0}{\sqrt{6m}} \\ \\ \dfrac{2mA_0}{\sqrt{3m}} - \dfrac{mA_0}{\sqrt{3m}} \end{pmatrix} = \begin{pmatrix} \dfrac{4mA_0}{\sqrt{6m}} \\ \\ \dfrac{mA_0}{\sqrt{3m}} \end{pmatrix}$$

$$\eta(t) = \begin{pmatrix} \frac{1}{\sqrt{6m}} \\ 2/\sqrt{6m} \end{pmatrix} \frac{4mA_0}{\sqrt{6m}} \cos\omega_1 t + \begin{pmatrix} 1/\sqrt{3m} \\ 1/\sqrt{3m} \end{pmatrix} \frac{mA_0}{\sqrt{3m}} \cos\omega_2 t$$

$$\eta(t) = \begin{pmatrix} \dfrac{4\cancel{m}A_0}{6\cancel{m}} \cos\omega_1 t + \dfrac{mA_0}{3m} \cos\omega_2 t \\ \\ \dfrac{8mA_0}{6m} \cos\omega_1 t + \dfrac{mA_0}{3m} \cos\omega_2 t \end{pmatrix}$$

$$\begin{pmatrix} \eta_1(t) \\ \eta_2(t) \end{pmatrix} = \begin{pmatrix} \dfrac{A_0}{3}\left(2\cos\omega_1 t + \cos\omega_2 t\right) \\ \\ \dfrac{A_0}{3}\left(4\cos\omega_1 t + \cos\omega_2 t\right) \end{pmatrix}$$

Problem 2 (10 Points):

An isolated uniform sphere of mass m and radius R is rotating with angular velocity ω_0 about an axis running through the sphere. Through only internal forces, the radius increases linearly to 2 R in a time τ, while maintaining uniform density and spherical symmetry.

a. At time τ, what is the angular velocity of the sphere? **(2 Points)**

b. Find an expression for the angular velocity as a function of time. **(1 Points)**

c. When the system reaches 2 R it immediately reverses and its radius linearly decreases to R over the period τ to 2τ. By what angle $\Delta\phi$ is the object behind in its rotation compared to a situation where the sphere does not expand between 0 and 2τ? **(4 Points)**

d. Consider the case where the radius of the sphere expands exponentially with some time constant τ_e. How much does the sphere rotate compared to the case where there is no expansion as t$\rightarrow \infty$? **(3 Points)**

2.

a) * Angular momentum conserved

$$I_0 \omega_0 = I(\tau)\omega(\tau)$$

$$\Rightarrow \omega(\tau) = \frac{I_0 \omega_0}{I_\tau} = \frac{\frac{2}{5} m R^2 \omega_0}{\frac{2}{5} m 4R^2} = \frac{1}{4}\omega_0$$

b) $$I_0 \omega_0 = I(t)\,\omega(t)$$

$$\frac{2}{5} m R_0^2 \omega_0 = \frac{2}{5} m \left(R_0 + \underset{\substack{\uparrow \\ \text{const}}}{v} t\right)^2 \omega(t)$$

$$\omega(t) = \frac{R_0^2}{\left(R_0 + \underset{\substack{\uparrow \\ \text{const}}}{v} t\right)^2}\,\omega_0$$

$\omega(t) =$

$$\text{At } t = \tau, \quad \omega(\tau) = \frac{1}{4}\omega_0$$

$$\frac{1}{4} = \frac{R_0^2}{(R_0 + v\tau)^2}$$

$$\Rightarrow R_0 + v\tau = 2R_0 \Rightarrow v = \frac{R_0}{\tau}$$

$$\omega_1(t) = \frac{R_0^2 \omega_0}{\left(R_0 + \frac{R_0}{\tau} t\right)^2} = \frac{\omega_0}{\left(1 + \frac{t}{\tau}\right)^2}$$

c) At $t = \tau$, $\omega_\tau = \frac{1}{4}\omega_0$, $R_\tau = 2R_0$

$$I_0 \omega_0 = I(t)\,\omega(t)$$

$$\Rightarrow \frac{2}{5} m\, 4R_0^2 \frac{1}{4}\omega_0 = \frac{2}{5} m (2R_0 - v t)^2 \omega(t)$$

$$\Rightarrow \omega_2(t) = \frac{R_0^2 \quad \omega_0}{(2R_0 - vt)^2}$$

At, $t = \tau$, $R = R_0$, $\omega_\tau = \omega_0$

$$\Rightarrow \quad 2R_0 - v\tau = R_0$$

$$\Rightarrow \quad v = \frac{R_0}{\tau}$$

$$\omega(t) = \frac{\omega_0}{\left(2 - \frac{t}{\tau}\right)^2}$$

$$\int_{\theta_0}^{\theta} d\theta = \int_0^{\tau} \omega_1(t)\,dt + \int_{\tau}^{2\tau} \omega_2(t)\,dt$$

$$\Delta\theta = \int_0^{\tau} \frac{\omega_0}{\left(1+\frac{t}{\tau}\right)^2} dt + \int_0^{\tau} \frac{\omega_0}{\left(2-\frac{t}{\tau}\right)^2} dt$$

$1 + \frac{t}{\tau} = x \Rightarrow dt = \tau\, dx$ | $2 - \frac{t}{\tau} = y \Rightarrow dt = -\tau\, dy$

$t = 0,\ x = 1$ | $t = \tau,\ x = 2$ | $t = 0,\ y = 2$ | $t = \tau,\ y = 1$

$$\theta = \tau \int_1^2 \frac{w_0}{x^2} dx + \tau \int_1^2 \frac{w_0}{y^2} dy$$

$$= \omega_0 \tau \left(-\frac{1}{x}\Big|_1^2 + -\frac{1}{y}\Big|_1^2 \right)$$

$$= \omega_0 \tau \left(-\frac{1}{2} + 1 - \frac{1}{2} + 1 \right)$$

$$= \omega_0 \tau$$

$$\theta' = \omega_0 2\tau$$

$$\Delta\theta = \theta' - \theta = \omega_0 \tau$$

d) $R(t) = R_0 e^{t/\tau_e}$

$$I_0 \omega_0 = I(t)\, \omega(t)$$

$$\omega(t) = \frac{\frac{2}{5} m R_0^2 \omega_0}{\frac{2}{5} m R_0^2 e^{2t/\tau}} = \omega_0 e^{-2t/\tau}$$

$$\theta = \int_0^t \omega_0 e^{-2t/\tau} dt = -\frac{\omega_0 \tau}{2} e^{-2t/\tau}\Big|_0^t$$

$$= \frac{\omega_0 \tau}{2}\left(1 - e^{-2t/\tau}\right)$$

$\theta = \frac{\omega_0 \tau}{2}$ as $t \to \infty$

F-1 0

Problem 3 (10 Points):

Consider the following Lagrangian

$$L = \left(\frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega^2 q^2\right) e^{2\gamma t}$$

assuming that $\omega > \gamma$ for the questions that follow.

a. Determine the Hamiltonian associated with this Lagrangian. **(3 Points)**

b. Find a transformation to new phase space variables that make H independent of time and show that these form a canonical transformation by determining a generating function of the form $F_2(q, P, t)$. **(4 Points)**

c. Using the equations of motion for the transformed Hamiltonian $K(Q, P, t)$, solve for $Q(t)$ and transform back to get $q(t)$. **(3 Points)**

3. $$L = \left(\tfrac{1}{2}m\dot{q}^2 - \tfrac{1}{2}m\omega^2 q^2\right)e^{2\gamma t}$$

assuming $\omega > \gamma$

a) $$L = \tfrac{1}{2}m\dot{q}^2 e^{2\gamma t} - \tfrac{1}{2}m\omega^2 q^2 e^{2\gamma t}$$

$$\Rightarrow \quad P_q = m\dot{q}\,e^{2\gamma t}$$

$$H = \dot{q}P_q - L$$

$$= \tfrac{1}{2}m\dot{q}^2 e^{2\gamma t} - \tfrac{1}{2}m\omega^2 q^2 e^{2\gamma t}$$

$$H = \frac{P_q^2}{2m}e^{-2\gamma t} + \tfrac{1}{2}m\omega^2 q^2 e^{2\gamma t}$$

The Lagrangian & Hamiltonian is time dependent

Hence $\frac{dH}{dt} \neq 0$, so the Hamiltonian is not a const of motion i.e. not conserved

b) $$H = \frac{p^2}{2m} e^{-2\gamma t} + \frac{1}{2} m\omega^2 q^2 e^{2\gamma t}$$

to make H independent of time we can make the following transformation

$$P = p e^{-\gamma t}$$

$$Q = \frac{q}{m\omega^2} e^{\gamma t}$$

For $F_2(q, P, t)$

$$p = \frac{\partial F_2}{\partial q} = P e^{+\gamma t} \Rightarrow F_2 = qPe^{\gamma t} + f(P)$$

$$Q = \frac{\partial F_2}{\partial P} = \frac{q}{m\omega^2} e^{\gamma t} \Rightarrow F_2 = \frac{qP}{m\omega^2} e^{\gamma t} + g(q)$$

$$F_2(q, P, t) = qPe^{\gamma t} + \frac{qP}{m\omega^2} e^{\gamma t}$$

$$= qPe^{\gamma t}\left(1 + \frac{1}{m\omega^2}\right)$$

$$\{Q, P\} = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q}$$

$$= \frac{e^{\gamma t}}{m\omega^2} e^{-\gamma t} - 0 = \frac{1}{m\omega^2}$$ $\Rightarrow$ So not a canonical transformation

but choosing $\underline{P} = p e^{-\gamma t}$

$Q = q e^{\gamma t}$

$\{Q, \underline{P}\} = e^{\gamma t} e^{-\gamma t} = 1 \Rightarrow$ so canonical transformation

$p = \frac{\partial F_2}{\partial q} = \underline{P} e^{+\gamma t} \Rightarrow F_2 = q \underline{P} e^{+\gamma t} + f(\underline{P})$

$Q = \frac{\partial F_2}{\partial \underline{P}} = q e^{\gamma t} \Rightarrow F_2 = q \underline{P} e^{\gamma t} + g(q)$

$F_2(q, \underline{P}, t) = q \underline{P} e^{\gamma t}$

c) $K(Q, \underline{P}, t) = H(q, p, t) + \frac{\partial F_2(q, \underline{P}, t)}{\partial t}$

$= \frac{p^2}{2m} e^{-2\gamma t} + \frac{1}{2} m\omega^2 q^2 e^{2\gamma t} + \gamma q \underline{P} e^{\gamma t}$

$= \frac{\underline{P}^2}{2m} + \frac{1}{2} m\omega^2 Q^2 + \gamma Q \underline{P}$

$\frac{dK}{dt} = 0$

$\dot{Q} = \frac{\partial K}{\partial \underline{P}} = \frac{\underline{P}}{m} + \gamma Q \Rightarrow \underline{P} = m\dot{Q} - m\gamma Q$

$\dot{\underline{P}} = -\frac{\partial K}{\partial Q} = -m\omega^2 Q - \gamma \underline{P}$

$$\dot{P} = -m\omega^2 Q - m\gamma\dot{Q} + m\gamma^2 Q$$

$$\Rightarrow \ddot{Q} = \frac{\dot{P}}{m} + \gamma\dot{Q}$$

$$= -\omega^2 Q - \cancel{\gamma\dot{Q}} + \gamma^2 Q + \cancel{\gamma\dot{Q}}$$

$$e^{i\sqrt{\omega^2-\gamma^2}}$$

$$\Rightarrow \ddot{Q} + (\omega^2 - \gamma^2) Q = 0$$

$$Q(t) = A e^{-\gamma t} \sin(\omega t + \alpha)$$

$$q(t) = A e^{-2\gamma t} \sin(\omega t + \alpha)$$

Problem 4 (10 Points):

The coffee purchased at rest stops is often too hot to drink. One way to cool off your coffee is to add ice, but how much ice should you add? Take the initial conditions for the coffee to be $T_0^{cof} = 80\,°\mathrm{C}$ and $V = 400\,\mathrm{ml}$. Take the initial conditions for the ice to be $T_0^{ice} = 0\,°\mathrm{C}$. The final temperature for the coffee and ice that you want to achieve is $T_f = 60\,°\mathrm{C}$. For the following questions assume that the coffee is pure water (a good assumption for most rest stop coffee) and the process is adiabatic with respect to the surroundings. Neglect volume changes of the coffee and ice and any temperature dependence of the heat capacity. The following thermodynamic properties of water may be useful:

$M = 18.0\,\mathrm{g\,mole^{-1}}$, molar mass

$\rho = 1.00\,\mathrm{g/cm^3}$, density

$\Delta H_{fus} = 6.00\,\mathrm{kJ\,mole^{-1}}$, heat of fusion

$C_p = 75.4\,\mathrm{J\,mole^{-1}\,K^{-1}}$, heat capacity of liquid

For parts (a.)-(c.) your answers should be in terms of the variables described here.

a. Find a general (algebraic solution) expression for the mass of ice, m, that is needed to cool the coffee to T_f? **(4 Points)**

b. Calculate, numerically, how many grams of ice you should add to your coffee to lower the temperature to $T_f = 60\,°\mathrm{C}$. **(1 Points)**

c. What is the entropy change of the system (coffee + ice)? Find an algebraic solution. **(3 Points)**

d. What is the entropy change of the surroundings? **(1 Points)**

e. Is this a thermodynamically reversible process? Explain. **(1 Points)**

4.

$T_0^{cof} = 80^\circ C = 353K$

$T_0^{ice} = 0^\circ C = 273K$

$T_f = 60^\circ C = 333K$

$V = 400 ml$

$M = 18 g mol^{-1}$

$\rho = 1 g/cm^3$

$\Delta H_{fus} = 6 kJ/mole$

$C_p = 75.4 \ J \ mole^{-1} K^{-1}$

→ Apparently, C_p & C_v is same for liquid as they are incompressible

a) $\Delta U = Q + W$ consider $\Delta V = 0$

$\Rightarrow \quad \Delta U = Q$

But, Heat capacity is

$$C = \frac{Q}{\Delta T}$$

and sp. heat capacity is

$$c = \frac{Q}{m \Delta T}$$

Thus, $Q = mc\Delta T$

$$\Delta U_{coffee} = m_W C_W \Delta T = m_W C_W (T_f - T_0^{cof})$$

$$\Delta U_{ice} = m_{ice} \Delta H_{fus} + m_{ice} C_W (T_f - T_0^{ice})$$

$\Delta U_{tot} = 0$

$$\Rightarrow \quad m_{ice} = - \frac{m_W C_W (T_f - T_0^{cof})}{\Delta H_{fus} + C_W (T_f - T_0^{ice})}$$

$\boxed{mL \rightarrow cm3}$

b) $m_W = \rho V = (1\,g/cm^3)(400\,cm^3) = 400g$

$$m_{ice} = - \frac{0.4 \times 75.4 (333 - 353)}{6 \times 10^3 + 75.4 (333 - 273)}$$

$$= 570g$$

c) $ds = \frac{Q}{T}$ const. volume $W = 0$

$$ds = \frac{dU}{T} = C_V \frac{dT}{T}$$

$$\Rightarrow \Delta S = C_V \int_{T_i}^{T_f} \frac{dT}{T} = C_V \ln \left(\frac{T_f}{T_i} \right)$$

$$\Delta S_{coffee} = C_V \ln \left(\frac{333}{353} \right)$$

$$\Delta S_{ice} = C_v \ln\left(\frac{T_f}{T_i}\right) + \frac{m_{ice}\,\Delta H_{fus}}{T_{ice}}$$

$$= C_v \ln\left(\frac{333}{273}\right) + \frac{.057 \times 6\times10^3}{273}$$

$$\Delta S_{(coffee+ice)} = \Delta S_{coffee} + \Delta S_{ice}$$

d) ice + coffe is adiabatic, so no heat transfer to the environment

$$đQ = 0$$

$$\therefore \quad \Delta S = 0$$

e) To be a reversible process the combined entropy of the system & the surrounding has to be zero. In this case it is not. So, <u>not</u> a Reversible process

Problem 5 (10 Points):

Consider a one dimensional ideal gas of electrons as a model for the conduction electrons in a one dimensional wire.

a. Determine the density of states g(E) for the one dimensional non-interacting electron system confined to a length, L. **(3 Points)**

b. What is the Fermi energy for this system? **(2 Points)**

c. What is the root mean square velocity of the electrons at $T = 0\,^{\circ}\mathrm{K}$? **(3 Points)**

d. What is the entropy of the electrons at $T = 0\,^{\circ}\mathrm{K}$? Justify your answer. **(2 Points)**

5.

a) For a electron in one dimension confined into a length L

$$\varepsilon_K = \frac{\hbar^2 K^2}{2m}$$

$$g(E) = 2\frac{1}{2\pi}\int dx\, dK\; \delta(\epsilon-\epsilon_K)$$

$$= \frac{2L}{2\pi}\int_{-\infty}^{+\infty} dK\; \delta(\epsilon-\epsilon_K)$$

$K=\infty$, $\epsilon_K=\infty$

$K=-\infty$, $\epsilon_K=-\infty$

Now, $K = \left(\frac{2m}{\hbar^2}\right)^{1/2} \epsilon_K^{1/2}$

$$\Rightarrow dK = \frac{1}{2}\left(\frac{2m}{\hbar^2}\right)^{1/2} \epsilon_K^{-1/2}\, d\epsilon_K$$

$$= \frac{2L}{2\pi}\,\frac{1}{2}\left(\frac{2m}{\hbar^2}\right)^{1/2}\int_{-\infty}^{+\infty} \epsilon_K^{-1/2}\,\delta(\epsilon-\epsilon_K)\, d\epsilon_K$$

$$g(\epsilon) = \frac{L}{2\pi}\left(\frac{2m}{\hbar^2}\right)^{1/2} \epsilon^{-1/2}$$

$$\Rightarrow \quad N = \int d\varepsilon \; g(\epsilon)\, f(\varepsilon)$$

$$\Rightarrow N = \frac{L}{2\pi}\left(\frac{2m}{\hbar^2}\right)^{1/2} \int d\varepsilon \; \epsilon^{-1/2} \frac{1}{e^{\beta(\epsilon-\mu)}+1}$$

At $T=0$, $\mu = \epsilon_F$ & $f(\epsilon) = \Theta(\epsilon_F - \epsilon)$

$\Rightarrow$

$$\Rightarrow N = \frac{L}{2\pi}\left(\frac{2m}{\hbar^2}\right)^{+1/2} \int_0^\infty d\varepsilon \; \varepsilon^{-1/2}\, \Theta(\varepsilon_F - \varepsilon)$$

$$\Rightarrow N = \frac{L}{2\pi}\left(\frac{2m}{\hbar^2}\right)^{1/2} \int_0^{\epsilon_F} \epsilon^{-1/2}\, d\epsilon$$

$$\Rightarrow N = \cancel{2}\frac{L}{\cancel{2}\pi}\left(\frac{2m}{\hbar^2}\right)^{1/2} \epsilon_F^{1/2}$$

$$\Rightarrow \epsilon_F = \left(\frac{\pi N}{L}\right)^2 \left(\frac{\hbar^2}{2m}\right)$$

c) $$\langle v^2 \rangle = \int v^2 \, g(\epsilon)\, f(\epsilon)\, d\epsilon \qquad \text{At } T=0,\; f(\epsilon) = \Theta(\epsilon_F - \epsilon)$$

$$= \frac{L}{2\pi}\left(\frac{2m}{\hbar^2}\right)^{1/2} \int v^2 \epsilon^{-1/2}\, \Theta(\epsilon_F - \epsilon)$$

$$\epsilon = \frac{1}{2} m v^2 \qquad = \frac{L}{2\pi}\left(\frac{2m}{\hbar^2}\right)^{1/2} \int_0^{\epsilon_F} \frac{2\epsilon}{m}\, \epsilon^{-1/2}\, d\epsilon$$

5.

$$\langle v^2 \rangle = \frac{L}{2\pi}\left(\frac{2m}{\hbar^2}\right)^{1/2} \frac{2}{m}\,\frac{2}{3}\,\epsilon^{3/2}\Big|_0^{\epsilon_F}$$

$$= \frac{L}{2\pi}\left(\frac{2m}{\hbar^2}\right)^{1/2} \frac{4}{3m}\left(\frac{\pi N}{L}\right)^3 \left(\frac{\hbar^2}{2m}\right)^{3/2}$$

$$V_{rms} = \sqrt{\langle v^2 \rangle} =$$

d. electrons indistinguishable at $T=0$

there is only one microstate

$$S = k \ln \Omega$$

so, $S \to 0$ as $T \to 0$

Problem 6 (10 Points):

The following questions refer to a stream of photons in equilibrium at temperature T (thermal light - say from a light bulb) incident on a perfect detector which detects (counts) all the particles that hit it. Your final answers should be in terms of the mean particle number.

a. Given $\bar{n}_s$ photons are counted on average in time t, calculate the variance in the photon number n_s, $\overline{(\Delta n_s)^2}$. **(2 Points)**

b. Calculate the fractional fluctuation of the detector signal defined as the square root of the variance divided by the mean photon number, $\bar{n}_s$, squared, $\sqrt{\overline{(\Delta n_s)^2}/\bar{n}_s^2}$. This is the inverse of the signal to noise ratio. **(2 Points)**

The following questions refer to a stream of electrons in equilibrium at temperature T incident on a detector which detects (counts) all the particles that hit it. Again, your final answers should be in terms of the mean particle number.

c. Given $\bar{n}_e$ electrons are counted on average in time t, calculate the variance in the electron number n_e, $\overline{(\Delta n_e)^2}$. **(2 Points)**

d. Calculate the fractional fluctuation of the detector signal defined as the square root of the variance divided by the mean electron number, $\bar{n}_e$, squared, $\sqrt{\overline{(\Delta n_e)^2}/\bar{n}_e^2}$. **(2 Points)**

e. Compare the two results. Are the results the same or different? Do the counts detected clump (bunch) or anti-clump (anti-bunch)? Why? **(2 Points)**

6.

$$\langle n_i \rangle = \sum_i n_i P_i = \frac{1}{Q} \sum_i n_i \, e^{-\beta(E-\mu) n_i}$$

$$= \frac{1}{Q} \sum_i n_i \, z \, e^{-\beta E n_i} \qquad z = e^{\beta\mu}$$

$$= -\frac{1}{Q} \frac{\partial}{\partial E} \sum_i z \, e^{-\beta E n_i}$$

$$= -\frac{1}{Q} \frac{\partial Q}{\partial E}$$

$$\langle n_i^2 \rangle = \sum_i n_i^2 P_i = \frac{1}{Q} \sum_i n_i^2 \, e^{-\beta(E-\mu) n_i}$$

$$= \frac{1}{Q} \sum_i n_i^2 \, z \, e^{-\beta E n_i}$$

$$= \frac{1}{Q} \frac{\partial^2}{\partial E^2} \sum_i z \, e^{-\beta E n_i}$$

$$= \frac{1}{Q} \frac{\partial^2 Q}{\partial E^2}$$

$$\overline{(\Delta n)^2} = \langle n_i^2 \rangle - \langle n_i \rangle^2$$

$$= \frac{1}{Q} \frac{\partial^2 Q}{\partial E^2} - \left(-\frac{1}{Q} \frac{\partial Q}{\partial E}\right)^2$$

$$= \frac{\partial^2}{\partial E^2} \ln Q$$

* For photons

$$Q = \prod_i \frac{1}{1-e^{-\beta E_i}} \qquad \mu = 0 \quad z = 1$$

$$\overline{(\Delta n)^2} = \frac{\partial^2}{\partial E^2}\left\{-\sum_i \ln\left(1-e^{-\beta E_i}\right)\right\}$$

$$= -\frac{\partial}{\partial E}\left\{\sum_i \frac{\beta e^{-\beta E_i}}{1-e^{-\beta E_i}}\right\}$$

$$= -\beta \frac{(1-e^{-\beta E_i})(-\beta E_i) - e^{-\beta E_i}(+\beta e^{-\beta E_i})}{(1-e^{-\beta E_i})^2}$$

$$= \sum_i \frac{\beta^2 e^{-\beta E_i}}{(1-e^{-\beta E_i})^2}$$

$$\langle n_s \rangle = -\frac{\partial}{\partial E} \ln \left\{\prod_i \frac{1}{1-e^{-\beta E_i}}\right\}$$

$$= \frac{\partial}{\partial E} \sum_i \ln\left(1-e^{-\beta E_i}\right)$$

$$= \sum_i \frac{\beta e^{-\beta E_i}}{1-e^{-\beta E_i}}$$

$$\frac{\beta}{\langle n \rangle} = \frac{1-e^{-\beta E_i}}{e^{-\beta E_i}} = e^{\beta E_i} - 1$$

$$e^{\beta E_i} = \frac{\beta}{\langle n \rangle} + 1$$

$$\sqrt{\frac{\overline{(\Delta n)^2}}{\overline{n_s}^2}} = \sqrt{e^{\beta E_i}} = \sqrt{\frac{\beta}{\langle n \rangle} + 1}$$

* For electrions

$$Q = \prod_i \left(1 + z e^{-\beta E_i}\right)$$

$$\langle n_s \rangle = -\frac{\partial}{\partial E} \ln \left\{ \prod_i \left(1 + z e^{-\beta E_i}\right) \right\}$$

$$= \sum_i \frac{z\beta e^{-\beta E_i}}{\left(1 + z e^{-\beta E_i}\right)}$$

$$\overline{(\Delta n)^2} = \frac{\partial^2}{\partial E^2} \left\{ \sum_i \ln\left(\left(1 + z e^{-\beta E_i}\right)\right) \right\}$$

$$= -\frac{\partial}{\partial E} \left\{ \frac{z\beta e^{-\beta E_i}}{\left(1 + z e^{-\beta E_i}\right)} \right\}$$

$$= -z\beta \frac{\left(1 + z e^{-\beta E_i}\right)\left(-\beta e^{-\beta E_i}\right) - e^{-\beta E_i}\left(-z\beta e^{-\beta E_i}\right)}{\left(\left(1 + z e^{-\beta E_i}\right)\right)^2}$$

$$= -z\beta \frac{-\beta e^{-\beta E_i}}{\left(1 + z e^{-\beta E_i}\right)^2}$$

$$= \frac{z\beta^2 e^{-\beta E_i}}{\left(1 + z e^{-\beta E_i}\right)^2}$$

$$\sqrt{\frac{\overline{(\Delta n)^2}}{\bar{n}}} = \sqrt{\frac{1}{z} e^{\beta E_i}}$$