Classical Mechanics and Statistical/Thermodynamics

August 2009

Possibly Useful Information

Handy Integrals:

$$\int_0^\infty x^n e^{-\alpha x}\,dx = \frac{n!}{\alpha^{n+1}}$$

$$\int_0^\infty e^{-\alpha x^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$$

$$\int_0^\infty x\ e^{-\alpha x^2}\,dx = \frac{1}{2\alpha}$$

$$\int_0^\infty x^2\ e^{-\alpha x^2}\,dx = \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}}$$

$$\int_{-\infty}^\infty e^{i\,a\,x-b\,x^2}\,dx = \sqrt{\frac{\pi}{b}}e^{-a^2/4b}$$

Geometric Series:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad \text{for} \quad |x| < 1$$

Stirling's approximation:

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

Riemann and related functions:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \equiv \zeta(p)$$

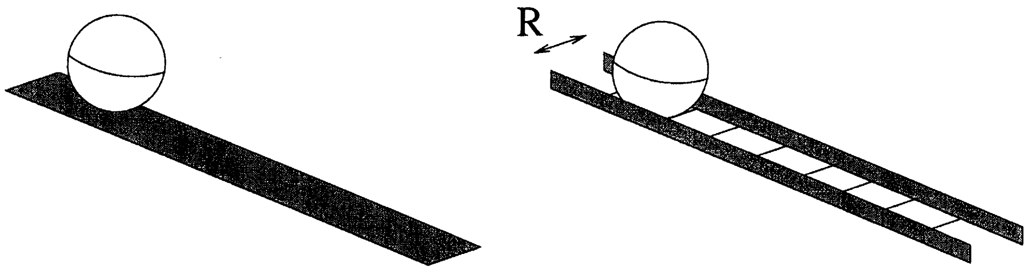
$$\sum_{n=1}^{\infty} \frac{z^p}{n^p} \equiv g_p(z) \qquad \sum_{n=1}^{\infty}(-1)^p\,\frac{z^p}{n^p} \equiv f_p(z)$$

$$g_p(1) = \zeta(p) \qquad f_p(1) = \zeta(-p)$$

$$\zeta(1) = \infty \qquad \zeta(-1) = 0.0833333$$
$$\zeta(2) = 1.64493 \qquad \zeta(-2) = 0$$
$$\zeta(3) = 1.20206 \qquad \zeta(-3) = 0.0083333$$
$$\zeta(4) = 1.08232 \qquad \zeta(-4) = 0$$

Classical Mechanics

1. Rolling spheres: Given that the moment of inertia of a sphere of mass m and radius R is $(2/5)mR^2$, please answer the following.



(a) A sphere of radius R and mass m rolls without slipping down an inclined plane on to a horizontal table (left figure). The condition of "rolling without slipping" forces a relationship between v, the speed of the center of mass of the sphere, and ω, the angular velocity of the sphere about its center of mass. What is this relationship? (0.5 pt.)

(b) Calculate the speed of the sphere at the bottom of the ramp if the center of mass of the sphere has dropped a distance h when it just touches the table. Assume that $h >> R$. (1.5 pt.)

(c) The ramp is now replaced by two narrow rails separated by a distance R (right figure). Again the ball rolls downward without slipping, supported by the two rails. In this case, what the relationship between v and ω? (1 pt.)

(d) In this second case, calculate the speed of the sphere at the bottom if the center of mass has dropped a distance h. (2 pts).

(e) After the ball reaches the bottom of the rails (part b) it continues to move on the horizontal table. It will either be rolling too fast or too slow to roll without slipping. Which will it be? You must prove your result. (1 pt.)

(f) Friction between the sphere and the plane will adjust the speed of the sphere until it can again roll without slipping. If the magnitude of the force of friction between the sphere and the plane is

μmg, determine the speed of the ball when it again rolls without slipping. (If you did not solve part (b) above, assume the sphere is moving at speed v_0 without rolling and determine its speed when it rolls without slipping). (4 pts.)

1.

a) $V = \omega R$

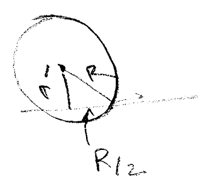
b) $mgh = \frac{1}{2} m v^2 + \frac{1}{2} I \omega^2$

$= \frac{1}{2} m v^2 + \frac{1}{2} \left(\frac{2}{5}\right) m R^2 \frac{V^2}{R^2}$

$= \left(\frac{1}{2} + \frac{1}{5}\right) m v^2$

$mgh = \frac{7}{10} m v^2$

$\Rightarrow \quad v_1 = \sqrt{\frac{10}{7} gh}$



c) $r' = \sqrt{R^2 - \frac{R^2}{4}} = \frac{\sqrt{3}R}{2}$

$v_2 = \omega r' = \frac{\sqrt{3}R\omega}{2}$

d) $mgh = \frac{1}{2} m v_2^2 + \frac{1}{2} I \omega^2 = \frac{1}{2} m v_2^2 + \frac{1}{2} \frac{2}{5} m R^2 \frac{4 v_2^2}{3R^2}$

$= \frac{1}{2} m v_2^2 + \frac{4}{15} m v_2^2 = \frac{23}{30} m v^2$

$\Rightarrow v_2 = \sqrt{\frac{30gh}{23}}$

e) $\omega_1 = \frac{v_1}{R} = \sqrt{10gh} / R$

$$\omega_2 = \frac{v_2}{\sqrt{3}R/2} = \sqrt{\frac{4\times30}{3\times23}gh} = \sqrt{\frac{40}{23}\frac{gh}{R}}$$

* it would be rolling too fast

f) when it will roll without slipping again it has to satisfy $v = \omega R$

$$v_f = v_i + at \qquad \omega_f = \omega_i + \alpha t$$

$F = ma$	$\tau = I\alpha$
$\Rightarrow -\mu mg = ma$	$\Rightarrow -\mu mg = \frac{2}{5}mR^2\alpha$ (cw)
$\Rightarrow a = -\mu g$	$\Rightarrow \alpha = -\frac{5\mu}{2R^2}$

$$\Rightarrow v_i - \mu g t = (\omega_i - \frac{5\mu}{2R^2}t)R$$

$$(\mu g - \frac{5\mu}{2R})t = v_i - \omega_i R$$

$$\Rightarrow t = \frac{2R}{2\mu gR - 5\mu}(v_i - \omega_i R)$$

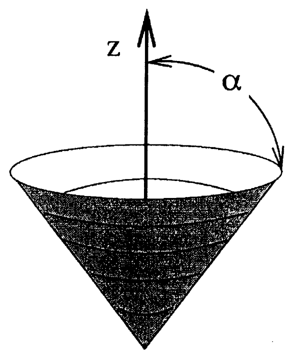
$$v_i - \omega_i R = \sqrt{\frac{30gh}{23}} - \sqrt{\frac{40}{23}gh\frac{1}{R}}R$$

$$= \frac{2R}{2\mu(gR - 5/2)}\sqrt{gh}\left(\frac{\sqrt{30}-\sqrt{40}}{\sqrt{23}}\right)$$

$$v = v_i - \mu g t$$

$$= \sqrt{\frac{30gh}{23}} - \frac{gR}{(gR-5/2)}\sqrt{gh}\left(\frac{\sqrt{30}-\sqrt{40}}{\sqrt{23}}\right)$$

2. A point particle of mass m travels on the frictionless inner surface of an inverted cone. The cone is oriented so its symmetry axis is parallel to the z-axis, with an opening angle α between the z-axis and the surface of the cone. The force of gravity points in the negative z-direction.



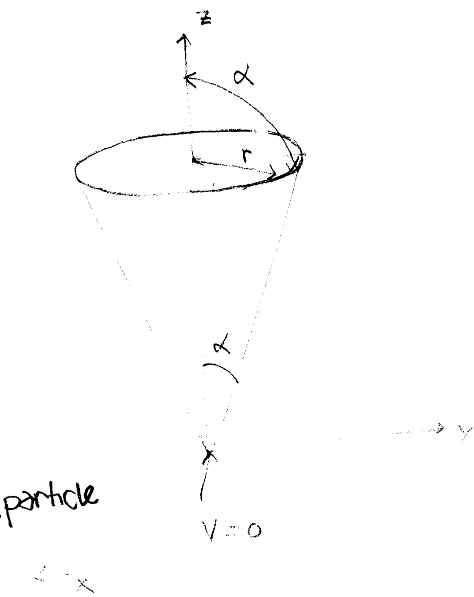
(a) Write the Lagrangian for the problem in cylindrical coordinates. (1 pt.)

(b) Assume the particle is moving in a uniform circular orbit at distance d from the cone tip, measured along the surface of the cone. Determine the angular frequency of the system. (3 pts.)

(c) The opening angle of the cone is abruptly decreased by $\Delta\alpha << \alpha$. This is done in a manner that does **not** impart an impulse or do work on the particle. (Imagine that the cone is instantaneously stretched so that its tip moves slightly downward, but the particle is not displaced during the stretching). Describe the subsequent motion of the particle in this limit. Express your answer in terms of ρ_0, the original radius of the circular orbit, m, α, $\Delta\alpha$, and g. Explain any approximations you are making in deriving your result. (6 pts.)

2.

a) $L = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) - mgz$



$x = r\cos\theta \quad y = r\sin\theta$

$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{2}m\dot{z}^2 - mgz$

b) * but r is constrained by its position on z-axis i.e. there is a constraint force on the particle

$\frac{r}{z} = \tan\alpha$

$\Rightarrow dr - \tan\alpha \, dz = 0$

$a_r = 1, \quad a_z = -\tan\alpha$

EOM with constraint force,

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = \lambda a_r$

$\Rightarrow m\ddot{r} - mr\dot{\theta}^2 = \lambda$ but $\ddot{r}=0 \Rightarrow \lambda = -mr\dot{\theta}^2$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = \lambda a_z$

$\Rightarrow m\ddot{z} - mg = \lambda\tan\alpha$

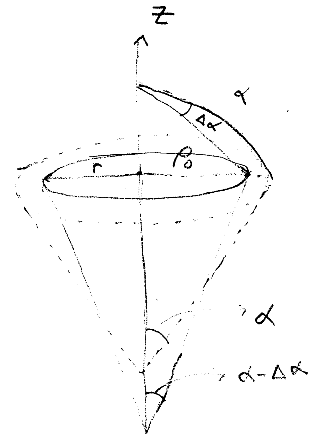
$\Rightarrow \lambda = -\frac{mg}{\tan\alpha}$

$mr\dot{\theta}^2 = \frac{mg}{\tan\alpha}$

$\dot{\theta} = \sqrt{\frac{mg}{\tan\alpha}} = \omega$

c)

The original, Lagrangian was



$$L = \frac{1}{2}m\dot{\rho}^2 + \frac{1}{2}m\rho^2\dot{\theta}^2 + \frac{1}{2}m\dot{z}^2 - mgz$$

$$\frac{\rho}{z}$$

3. Consider the Lagrangian for a 1D system with generalized coordinate q:

$$L(q, \dot{q}, t) = e^{\lambda t/m} \left[\frac{m}{2}\dot{q}^2 - \frac{m\omega_0^2}{2} q^2 \right] \qquad (1)$$

In the above expression, m is a mass, ω_0 is a frequency, and λ is a positive and dimensionless constant.

(a) Derive the equation of motion for the system. (1 pt.)

(b) What is the canonical momentum, p? (1 pt.)

(c) Calculate the Hamiltonian. (3 pts.)

(d) We wish to make a canonical transformation $(q, p) \rightarrow (Q, P)$ using the generating function

$$F_2(q, P, t) = e^{\lambda t/2m} q\, P$$

What is the new coordinate and canonical momentum in terms of the old? (2 pts.)

(e) Show that the canonically transformed Hamiltonian is not time dependent. (3 pts.)

Statistical Mechanics

4. It can be shown that the Helmholtz free energy for a photon gas is given by:

$$F(T, V, N) = -\frac{1}{3}\sigma V T^4$$

where σ is the Stefan-Boltzmann constant. Using this relation, answer the following:

(a) What are the equations of state (that is, P, S, and μ as functions of T, V and N)? (3pts.)

(b) Consider a Carnot cycle using a photon gas as its working fluid. The cycle is driven by one hot and one cold temperature reservoir, with temperatures T_h and T_c respectively. Draw the cycle in the P-V plane. **Caution:** This is **not** an ideal gas! Think carefully about the steps in a Carnot cycle and use your results from above to determine what the cycle will look like. (2pts.)

(c) Solve for the heat exchanged in each leg of your Carnot cycle. Your answer may depend upon T_h, T_c, and any other variables you might choose in defining your cycle. (2pts.)

getting two diff ans

(d) Using these values for the heat exchanged, calculate the efficiency of a Carnot cycle that uses a photon gas as its working fluid. If you cannot calculate it, devise a careful argument for its value. (3pts.)

4.

a) $F = E - TS$

$\Rightarrow dF = dE - TdS - SdT$

$\Rightarrow dF = \underbrace{dQ}_{= T\not{dS}} - PdV + \mu dN - T\not{dS} - SdT$

$\Rightarrow -\frac{1}{3}\sigma T^4 dV - \frac{4}{3}\sigma V T^3 dT = -PdV - SdT + \mu dN$

$P = \frac{1}{3}\sigma T^4$

$S = \frac{4}{3}\sigma V T^3$

$\mu = 0$

b) * carnot cycle has two isotherm & two adiabate

→ Adiabatic : $dS = \frac{4}{3}\sigma\left(T^3 dV + 3VT^2 dT\right) = 0$

$\Rightarrow dT = -\frac{T}{3V} dV$

$dP = \frac{4}{3}\sigma T^3 dT = -\frac{4}{9V}\sigma T^4 dV$

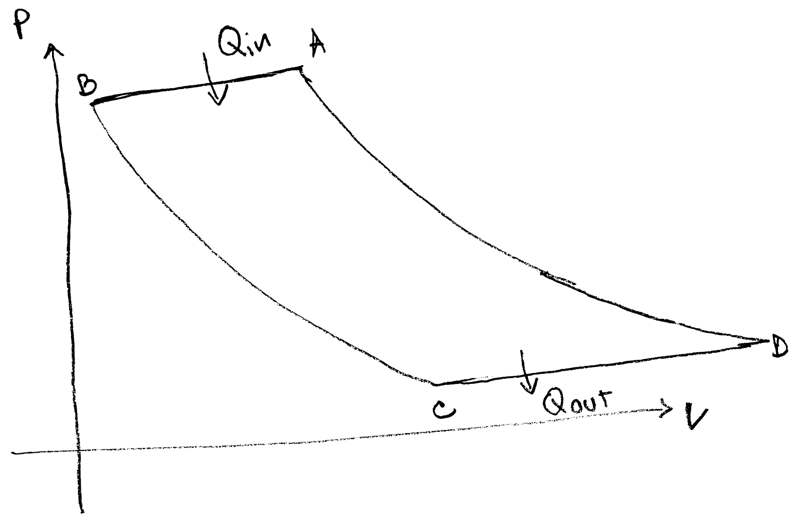
$dP = -\frac{4}{3}\frac{P}{V} dV$

$\Rightarrow \ln P = \ln V^{-4/3} \Rightarrow P = V^{-4/3}$

→ Isotherm:

$$dP = \frac{4}{3}\sigma T^3 \underbrace{dT}_{0} = 0$$

P is independent of T & V in isotherm,



c) Isotherm,

$$\Delta E = 0$$

$$Q_{in} = -W = \int P\,dV = P(V_B - V_A) = \frac{1}{3}\sigma T^4 (V_B - V_A)$$

$$ds = \frac{đQ}{T_h} \Rightarrow Q = \int ds\, T_h$$

* two diff ans

$$= \frac{4}{3}\sigma T_h^4 (V_B - V_A)$$

$$= \frac{4}{3}\sigma T_h^4 (V_B - V_A)$$

5. A particular solid is made up of N distinguishable spin 1 atoms each on a fixed position in a lattice. The energy of each atom is given by:

$$E(\sigma_i) = -V_0\sigma_i^2 - \mu_0\sigma_i B$$

where V_0 arises from an internal field in the crystal, B is the applied external magnetic field and μ_0 is the Bohr magneton. The z-component of the spin of an atom can take on values $\sigma_i \in \{0, \pm 1\}$

   (a) Calculate the free energy, $F(T, B, N)$. (2 pts.)

   (b) Calculate the specific heat. (4 pts.)

   (c) Calculate the magnetic susceptibility, $\chi(T, B, N)$ when $B = 0$. (4 pts.)

The one particle energy is

$$\epsilon(\sigma_i) = -V_0\sigma_i^2 - \mu_0\sigma_i B$$

one particle partion func then would be

$$Z_1 = \sum_{\sigma_i} e^{-\beta\epsilon(\sigma_i)} = \sum_{\sigma_i} e^{+\beta(V_0\sigma_i^2+\mu_0\sigma_i B)}$$

Since they are distinguishable & non-interacting the N-particle partition func is

$$Z = Z_1^N = \sum_{\sigma_i} e^{-N\beta\epsilon(\sigma_i)}$$

a) $$F = -KT\ln Z$$

$$= -NKT\ln\left(\sum_{\sigma_i} e^{+\beta(V_0\sigma_i^2+\mu_0\sigma_i B)}\right)$$

$$= -NKT\ln\left(1+e^{\beta(V_0+\mu_0 B)}+e^{\beta(V_0-\mu_0 B)}\right)$$

$$= -NKT\ln\left[e^{\beta V_0}\left(e^{\beta\mu_0 B}+e^{-\beta\mu_0 B}\right)+1\right]$$

$$= -NKT\ln\left[e^{\beta V_0}2\cosh(\beta\mu_0 B)+1\right]$$

b)

$$F = E - TS$$

$$dF = dE - Tds - SdT$$

$$= T\,ds + PdV - T\,ds - SdT + \mu dN$$

$$= PdV - SdT + \mu dN \qquad S = -\frac{\partial F}{\partial T}\bigg|_{N,V}$$

$$E = F + TS$$

$$= F - T\frac{\partial F}{\partial T}$$

$$= -KT\ln Z - T\frac{\partial}{\partial T}(-KT\ln Z)$$

$$= -KT\ln Z + KT\ln Z + KT^2\frac{\partial \ln Z}{\partial T}$$

$$= KT^2\frac{\partial \ln Z}{\partial \beta}\,\frac{\partial \beta}{\partial T}$$

$$= KT^2\frac{\partial \ln Z}{\partial \beta}\left(-\frac{1}{KT^2}\right)$$

$$= -\frac{\partial \ln Z}{\partial \beta}$$

$$E = -N\frac{\partial}{\partial\beta}\ln\left[2e^{\beta V_0}\cosh(\beta\mu_0 B)+1\right]$$

$$= -N\frac{2V_0e^{\beta V_0}\cosh(\beta\mu_0 B)+2\mu_0 B\,e^{\beta V_0}\sinh(\beta\mu_0 B)}{1+2e^{\beta V_0}\cosh(\beta\mu_0 B)}$$

$$C_V = \frac{\partial E}{\partial T}$$

$$= \frac{\partial E}{\partial\beta}\frac{\partial\beta}{\partial T}$$

$$= -\frac{1}{KT^2}\frac{\partial E}{\partial\beta}$$

$$= \frac{N}{KT^2}\left\{\frac{v\left\{2V_0e^{\beta V_0}\left(V_0\cosh()+\mu_0 B\sinh()\right)+2e^{\beta V_0}\left(V_0\mu_0 B\sinh()+\mu_0^2B^2\cosh()\right)\right.}{\left(1+2e^{\beta V_0}\cosh(\beta\mu_0 B)\right)^2}\right.$$

$$\left. - \frac{u\,2\left(1+2e^{\beta V_0}\cosh()\right)\left(2V_0e^{\beta V_0}\cosh()+2\mu_0 Be^{\beta V_0}\sinh()\right)\}}{v^2}\right\}$$

$$= \frac{N}{KT^2}\frac{2ve^{\beta V_0}\left((V_0^2+\mu_0^2B)^2\cosh()+2V_0\mu_0 B\sinh()\right)-2u}{\left(1+2e^{\beta V_0}\cosh()\right)^2}$$

will not simplify any further

(c) * Susceptibility is defined as

$$\bar{M} = \chi \bar{B}$$

↑ magnetization

* Magnetization is the alignment of spin along the direction of the $\bar{B}$-field. Notice this is a bound energy. So we can evaluate magnetization by calculating the variation of the free energy with magnetic field

$$\langle \bar{M} \rangle = -\frac{\partial F}{\partial B} = KT \frac{\partial \ln Z}{\partial \beta} = \frac{1}{\beta} \frac{\partial \ln Z}{\partial \beta}$$

↑ since bound energy

$$M = \frac{N}{\beta} \frac{2e^{\beta V_0}\left[V_0 \cosh(\beta \mu_0 B) + \mu_0 B \sinh(\beta \mu_0 B)\right]}{1 + 2^{\beta V_0} \cosh(\beta \mu_0 B)}$$

Now, calculatin χ is straight forward by def,

$$\chi = \frac{\partial M}{\partial B}\Big|_{B=0}$$