Classical Mechanics and Statistical/Thermodynamics

August 2008

Possibly Useful Information

Handy Integrals:

$$\int_0^\infty x^n e^{-\alpha x}\, dx = \frac{n!}{\alpha^{n+1}}$$

$$\int_0^\infty e^{-\alpha x^2}\, dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$$

$$\int_0^\infty x\ e^{-\alpha x^2}\, dx = \frac{1}{2\alpha}$$

$$\int_0^\infty x^2\ e^{-\alpha x^2}\, dx = \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}}$$

$$\int_{-\infty}^\infty e^{i\,a\,x-b\,x^2}\, dx = \sqrt{\frac{\pi}{b}}e^{-a^2/4b}$$

Geometric Series:

$$\sum_{n=0}^\infty x^n = \frac{1}{1-x} \qquad \text{for} \quad |x|<1$$

Stirling's approximation:

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

Riemann and related functions:

$$\sum_{n=1}^\infty \frac{1}{n^p} \equiv \zeta(p)$$

$$\sum_{n=1}^\infty \frac{z^p}{n^p} \equiv g_p(z) \qquad \sum_{n=1}^\infty (-1)^p \frac{z^p}{n^p} \equiv f_p(z)$$

$$g_p(1) = \zeta(p) \qquad f_p(1) = \zeta(-p)$$

$$\zeta(1) = \infty \qquad \zeta(-1) = 0.0833333$$

$$\zeta(2) = 1.64493 \qquad \zeta(-2) = 0$$

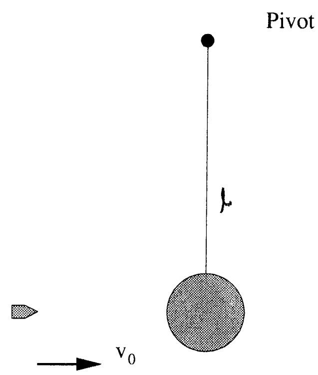
$$\zeta(3) = 1.20206 \qquad \zeta(-3) = 0.0083333$$

$$\zeta(4) = 1.08232 \qquad \zeta(-4) = 0$$

F-08

Classical Mechanics

1. **The ballistic pendulum:** Consider a pendulum with a bob of mass m connected to a frictionless pivot by an ideal massless rigid rod of length ℓ. A projectile of mass ϵm ($0 < \epsilon << 1$) moving horizontally at speed v_0 hits the center of the bob, as shown. When it strikes, it becomes imbedded in the bob.



   (a) What is the minimum initial speed of the projectile such that the pendulum will make a full rotation? (2 points)

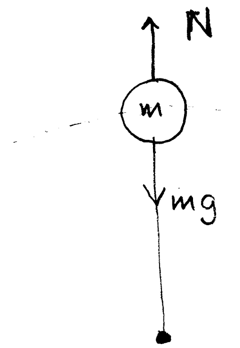
   (b) The rod is replaced by an ideal massless non-rigid string. What is the minimum initial speed of the projectile such that the pendulum will make a full revolution without the string going slack? (3 points)

   (c) Now assume that projectile rebounds elastically from the bob in the horizontal direction. What is the minimum initial speed of the projectile such that the pendulum will make a full revolution without the string going slack? (2 points)

   (d) Finally, assume that the projectile passes completely through the pendulum bob, in a time $t << \sqrt{\ell/g}$. After it exits, it carries with it some of the original mass of the bob, such that the exiting projectile now has a mass $2\epsilon m$ and moves at a speed $3v_0/4$. What is the minimum initial speed of the projectile such that the pendulum will make a full revolution without the string going slack? (3 points)

1.

a) Rigid rod, at the top



$F_{Net} = 0 \; , \; v = 0$

* the bullet sticks inside the bob $\Rightarrow$ inelastic collision

$\rightarrow$ conservation of momentum

$$\epsilon m v_0 = (m + \epsilon m) v$$

$$\Rightarrow v_0 = \left(1 + \frac{1}{\epsilon}\right) v$$

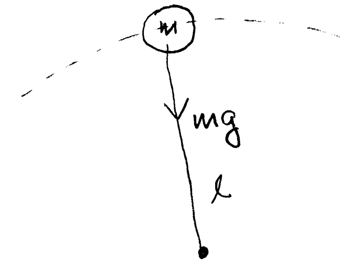
$\longrightarrow$ Using conservation of energy on the bob

$$\frac{1}{2}(m + \epsilon m) v^2 = (m + \epsilon m) g \, 2l$$

$$\Rightarrow v = 2\sqrt{gl}$$

$$\Rightarrow v_0 = 2\sqrt{gl}\left(1 + \frac{1}{\epsilon}\right)$$

b) non-rigid rod



At the top it is moving in a circular path

so, $\cancel{m}g = \frac{\cancel{m} v_{top}^2}{l}$

$$\Rightarrow v_{top}^2 = gl$$

Using conservation of momentum

$$v_0 = \left(1+\frac{1}{\epsilon}\right) v_{bottom}$$

Knowing v_{top} use conservation of energy

$$\frac{1}{2}(\cancel{m+\epsilon m}) v_{bottom}^2 = (\cancel{m+\epsilon m}) g (2l) + \frac{1}{2}(\cancel{m+\epsilon m}) v_{top}^2$$

$$\Rightarrow v_{bottom}^2 = 4gl + gl$$

$$\Rightarrow v_{bottom} = \sqrt{5gl}$$

$$\therefore v_0 = \left(1+\frac{1}{\epsilon}\right)\sqrt{5gl}$$

c) * Elastic collision

$$P_i = P_f$$

$$\Rightarrow \epsilon m v_0 = \epsilon m v_1 + m v_2 \Rightarrow v_0 = v_1 + \frac{1}{\epsilon} v_2$$

$$\Rightarrow v_1 = v_0 - \frac{1}{\epsilon} v_2$$

$$E_i = E_f$$

$$\Rightarrow \frac{1}{2}\epsilon m v_0^2 = \frac{1}{2}\epsilon m v_1^2 + \frac{1}{2} m v_2^2$$

$$\Rightarrow v_0^2 = \left(v_0 - \frac{1}{\epsilon} v_2\right)^2 + \frac{1}{\epsilon} v_2^2$$

$$\Rightarrow \cancel{v_0^2} = \cancel{v_0^2} + \frac{1}{\epsilon^2} v_2^2 - \frac{2 v_0 v_2}{\epsilon} + \frac{1}{\epsilon} v_2^2$$

$$\Rightarrow v_0 = \frac{1}{2}\epsilon v_2 + \frac{1}{2} v_2 = \frac{1}{2} v_2 (1+\epsilon)$$

Now, $v_2 = v_{\text{bottom pendulum}}$

but, $mg = \frac{m v_{top}^2}{l}$

$$\Rightarrow v_{top}^2 = g l$$

$$\frac{1}{2} m v_2^2 = \frac{1}{2} m v_{top}^2 + mg(2l)$$

$$v_2^2 = gl + 4gl = 5gl$$

$$v_0 = \frac{1}{2}(1+\epsilon)\sqrt{5gl}$$

d) * Momentum conservation

$$\epsilon m v_0 = (m-\epsilon m) v_1 + 2\epsilon m v_2$$

$$\Rightarrow \quad v_0 = \frac{(1-\epsilon) v_1}{\epsilon} + 2 v_2$$

but $v_2 = \frac{3v_0}{4}$

$$\Rightarrow \quad v_0 = \left(\frac{1}{\epsilon}-1\right) v_1 + \frac{3}{2} v_0$$

$$\Rightarrow \quad -\frac{1}{2} v_0 = \left(\frac{1}{\epsilon}-1\right) v_1$$

$$\Rightarrow \quad v_0 = 2\left(1-\frac{1}{\epsilon}\right) v_1$$

Now,

$$v_1 = v_{bottom} = \sqrt{5gl}$$

$$v_0 = 2\left(1-\frac{1}{\epsilon}\right)\sqrt{5gl}$$

2. The isotropic harmonic oscillator.

   (a) Write the Lagrangian for a point mass m moving under the influence of an isotropic 3-dimensional harmonic oscillator potential

   $$V(x, y, z) = \frac{k}{2}(x^2 + y^2 + z^2).$$

   There is no external gravitational field. (1 point)

   (b) Using the Lagrange equations of motion show that angular momentum is conserved. i.e.,

   $$\frac{d}{dt}\mathbf{L} = \frac{d}{dt}(\mathbf{r} \times m\mathbf{v}) = 0.$$

   Because the Lagrangian is invariant under rotations about the origin, you can choose coordinates so that motion is constrained to the x-y plane, i.e., the angular momentum points in the z direction. (3 points)

   (c) For 2-dimensional motion in the x-y plane choose cylindrical polar coordinates and proceed to solve the Lagrange equations of motion. You can leave the solution for $r(t)$ as an integral of the form $t = \int f(r)dr$. (Don't forget to use conservation of energy, E_0.) (3 points)

   (d) Compute the minimum and maximum values or the radial coordinate r as functions of the constants m, E_0, k, L^z. (3 points)

2.

a) $$L = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) - \frac{k}{2}(x^2+y^2+z^2)$$

$$x = r\cos\phi\sin\theta$$

$$y = r\sin\phi\sin\theta$$

$$z = r\cos\theta$$

$$\rightarrow \dot{x} = \dot{r}\cos\phi\sin\theta - r\sin\phi\sin\theta\dot{\phi} + r\cos\phi\cos\theta\dot{\theta}$$

$$\rightarrow \dot{y} = \dot{r}\sin\phi\sin\theta + r\cos\phi\sin\theta\dot{\phi} + r\sin\phi\cos\theta\dot{\theta}$$

$$\rightarrow \dot{z} = \dot{r}\cos\theta - r\sin\theta\dot{\theta}$$

$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{2}mr^2\sin^2\theta\dot{\phi}^2 - \frac{k}{2}r^2$$

b) trivial

c) $L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - \frac{K}{2}r^2$

$$m\ddot{r} - mr\dot{\theta}^2 + \frac{K}{2} = 0$$

$$mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} = 0$$

$$\Rightarrow r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0$$

$$\Rightarrow r\frac{d\dot{\theta}}{dr}\cancel{\dot{r}} = -2\cancel{\dot{r}}\dot{\theta}$$

$$\Rightarrow \frac{d\dot{\theta}}{\dot{\theta}} = -2\frac{dr}{r}$$

$$\Rightarrow \ln\dot{\theta} = -2\ln r + \text{const}$$

$$\Rightarrow \dot{\theta} = \frac{1}{r^2}\text{const}$$

$$\rightsquigarrow m\ddot{r} - mr\frac{A}{r^4} + \frac{K}{2} = 0$$

$$\Rightarrow m\ddot{r} = +\frac{mA}{r^3} - \frac{K}{2}$$

$$\Rightarrow m\frac{d\dot{r}}{dr}\dot{r} = \frac{mA}{r^3} - \frac{K}{2}$$

$\Rightarrow$ easier $\Rightarrow$

$$\Rightarrow m\dot{r}\,d\dot{r} = \left(\frac{mA}{r^3} - \frac{K}{2}\right)dr$$

In cylindrical co-ordinate

c) $$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + \frac{k}{2}r^2$$

* To evaluate the eqns of motion it is actually easier to work with the Hamiltonian and do Legendre transformation where the Hamiltonian is a const of motion and write it down as a Hamilton - Jacobi eqn

$$H(q,p) = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{k}{2}r^2$$

θ cyclic $\quad \dot{p}_\theta = 0 \Rightarrow p_\theta = \alpha_\theta = \text{const.}$ $\downarrow$ angular momentum

* Use Hamilton's principle func $S(q,\underset{\downarrow}{\overset{P}{\alpha}},t)$ as a generating func to get the new Hamiltonian $(K=0)$

* Our H is independent of time

so, $$S(q,\alpha,t) = W(q,\alpha) - Et$$
$\uparrow$ Hamiltons characteristic func

$$W(q,\alpha) = W_r + \alpha_\theta \theta$$

$$p_r = \frac{\partial W_r}{\partial r}$$

So,

$$\frac{1}{2m}\left(\frac{dW_r}{dr}\right)^2 + \frac{1}{2mr^2}\alpha_\theta^2 + \frac{k}{2}r^2 = E$$

$$\Rightarrow \quad W_r = \int \left(2mE - \frac{\alpha_\theta^2}{r^2} - 2mkr^2\right)^{1/2} dr$$

$$\Rightarrow \quad R = \frac{\partial W}{\partial P_r} = \frac{\partial}{\partial E}\int (\quad)^{1/2} dr$$

$$\text{but} \quad \dot{R} = \frac{\partial H}{\partial P_r} = \frac{\partial E}{\partial E} = 1$$

$$R = t - t_0$$

$$\Rightarrow \quad t - t_0 = \int \frac{2m}{\sqrt{2mE - \frac{L^2}{r^2} - 2mkr^2}} dr$$

(d) $$E = \frac{1}{2} m \dot{r}^2 + \underbrace{\frac{L^2}{2mr^2} + \frac{K}{2} r^2}_{V_{eff}}$$

$$\dot{r}^2 = \frac{2E}{m} - \frac{L^2}{m^2 r^2} - \frac{K}{m} r^2$$

At the turning pts, $\dot{r} = 0$

$$\Rightarrow \quad r^4 - \frac{2E}{K} r^2 + \frac{L^2}{mK} = 0$$

$$\frac{\cancel{4}L^2}{m\cancel{K}} \times \frac{\cancel{K^2}}{\cancel{4}E^2}$$

$$\Rightarrow \quad r^2 = \frac{(2E/K) \pm \sqrt{(2E/K)^2 - 4L^2/mK}}{2}$$

$$= \frac{E}{K} \left[1 \pm \sqrt{1 - \frac{1}{m} \left(\frac{L}{E} \right)^2} \right]$$

$$\therefore \quad r = + \sqrt{\frac{E}{K}} \left[1 \pm \sqrt{1 - \frac{1}{m} \left(\frac{L}{E} \right)^2} \right]^{1/2}$$

$$r_{max} = \sqrt{\frac{E}{K}} \left[1 + \sqrt{1 - \frac{1}{m} (L/E)^2} \right]^{1/2}$$

$$r_{min} = \sqrt{\frac{E}{K}} \left[1 - \sqrt{1 - \frac{1}{m} (L/E)^2} \right]$$

3. Consider a particle attracted by a fixed gravitating body while also in a uniform gravitational field oriented along the z-axis. The potential energy is of the form:

$$V(r, z) = -m\left(\frac{k}{r} + g\,z\right)$$

where m is the particle's mass, k and g are constants, and r is the standard radial coordinate:

$$r \equiv \sqrt{x^2 + y^2 + z^2}$$

You are to examine the problem in *cylindrical parabolic coordinates* defined by

$$\begin{aligned} \zeta &\equiv r + z \\ \eta &\equiv r - z \\ \phi &\equiv \arctan y/x \end{aligned}$$

In these coordinates we may write the Cartesian coordinates as:

$$\begin{aligned} x &= \sqrt{\zeta\eta}\cos\phi \\ y &= \sqrt{\zeta\eta}\sin\phi \\ z &= \tfrac{1}{2}(\zeta - \eta) \end{aligned}$$

(a) Show that the kinetic energy, T, is given by:

$$T = \frac{m}{8}\left[\left(1 + \frac{\zeta}{\eta}\right)\dot{\eta}^2 + \left(1 + \frac{\eta}{\zeta}\right)\dot{\zeta}^2\right] + \frac{m}{2}\,\zeta\eta\,\dot{\phi}^2$$

in these coordinates. (2 points)

(b) What are the canonical momenta, p_ζ, p_η, and p_ϕ, expressed in cylindrical parabolic coordinates? (2 points)

(c) Use Hamilton-Jacobi theory to find the constants of the motion. *Hint:* While the total energy E does not separate in these coordinates, $E(\zeta + \eta)$ can be used to produce a quantity that **does** seperate. (3 points)

(d) What is Hamilton's characteristic function associated with ϕ? (1 point)

(e) Express Hamilton's characteristic functions associated with ζ, η as definite integrals. (2 points)

3. In cartesian co-ordinate

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + m \left(\frac{K}{\sqrt{x^2+y^2+z^2}} + gz \right)$$

In terms of cylindrical parabolic co-ordinate

$$X = \sqrt{\xi \eta} \cos\phi$$

$$Y = \sqrt{\xi \eta} \sin\phi$$

$$Z = \frac{1}{2} (\xi - \eta)$$

$$r^2 = x^2 + y^2 + z^2 = \xi\eta + \frac{1}{4} (\xi - \eta)^2 = \frac{1}{4} (\xi^2 + \eta^2) + \frac{1}{2} \xi\eta)$$
$$= \frac{1}{4} (\xi + \eta)^2$$

$$\dot{x} = - \sqrt{\xi\eta} \sin\phi \, \dot{\phi} + \frac{1}{2} \eta (\xi\eta)^{-1/2} \cos\phi \, \dot{\xi} + \frac{1}{2} \xi (\xi\eta) \cos\phi \, \dot{\eta}$$

$$\dot{y} = \sqrt{\xi\eta} \cos\phi \, \dot{\phi} + \frac{1}{2} \eta (\xi\eta)^{-1/2} \sin\phi \, \dot{\xi} + \frac{1}{2} \xi (\xi\eta)^{-1/2} \sin\phi \, \dot{\eta}$$

$$\dot{z} = \frac{1}{2} (\dot{\xi} - \dot{\eta})$$

$$\rightsquigarrow \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \xi\eta \dot{\phi}^2 + \frac{1}{4} \eta^2 (\xi\eta)^{-1} \dot{\xi}^2 + \frac{1}{4} \dot{\xi}^2 + \frac{1}{4} \xi^2 (\xi\eta)^{-1} \dot{\eta}^2 + \frac{1}{4} \dot{\eta}^2$$

$$= \frac{1}{4} \left(1 + \frac{\xi}{\eta}\right) \dot{\eta}^2 + \frac{1}{4} \left(1 + \frac{\eta}{\xi}\right) \dot{\xi}^2 + \xi\eta \dot{\phi}^2$$

* the mixed terms cancel each other

$$T = \frac{m}{8}\left[\left(1+\frac{\xi}{\eta}\right)\dot{\eta}^2 + \left(1+\frac{\eta}{\xi}\right)\dot{\xi}^2\right] + \frac{m}{2}\xi\eta\dot{\phi}^2$$

* no velocity term in potential

b)

$$p_\xi = \frac{\partial T}{\partial \dot{\xi}} = \frac{m}{4}\left(1+\frac{\eta}{\xi}\right)\dot{\xi}$$

$$p_\eta = \frac{\partial T}{\partial \dot{\eta}} = \frac{m}{4}\left(1+\frac{\xi}{\eta}\right)\dot{\eta}$$

* ϕ cyclic

$$p_\phi = \frac{\partial T}{\partial \dot{\phi}} = m\xi\eta\dot{\phi}$$

$\dot{p}_\phi = 0, \quad p_\phi = \alpha_\phi = \text{const}$

c)

$$V(r,z) = -m\left(\frac{k}{r} + gz\right)$$

$$= -m\left(\frac{2k}{\xi+\eta} + \frac{g}{2}(\xi-\eta)\right)$$

$$H = \dot{\xi}p_\xi + \dot{\eta}p_\eta + \dot{\phi}p_\phi - L$$

$$= \frac{m}{8}\left(1+\frac{\xi}{\eta}\right)\dot{\eta}^2 + \frac{m}{8}\left(1+\frac{\eta}{\xi}\right)\dot{\xi}^2 + \frac{m}{2}\xi\eta\dot{\phi}^2 + \frac{2mk}{(\xi+\eta)} - \frac{mg}{2}(\xi-\eta)$$

$$= \frac{2p_\eta^2}{m\left(1+\frac{\xi}{\eta}\right)} + \frac{2p_\xi^2}{m\left(1+\frac{\eta}{\xi}\right)} + \frac{\alpha_\phi^2}{2m\xi\eta} + \frac{2mk}{\xi+\eta} - \frac{mg}{2}(\xi-\eta)$$

Using $W(q,\alpha)$ as the generating func

$$p_\eta = \frac{\partial W_\eta}{\partial \eta}, \quad p_\xi = \frac{\partial W_\xi}{\partial \xi}, \quad p_\phi = \alpha_\phi = \frac{\partial W_\phi}{\partial \phi}$$

* In this case, $H = E$

$$E = \frac{2}{m(\xi+\eta)}\left\{\eta\left(\frac{\partial W_\eta}{\partial \eta}\right)^2 + \xi\left(\frac{\partial W_\xi}{\partial \xi}\right)^2 + 2m^2K\right\}$$

$$+\frac{1}{2m\xi\eta}\left(\frac{\partial W_\phi}{\partial \phi}\right)^2 - \frac{mg}{2}(\xi-\eta)$$

$$\Rightarrow \frac{mE}{2}(\xi+\eta) = \{\quad\} + \frac{1}{4}\frac{\xi+\eta}{\xi\eta}\left(\frac{\partial W_\phi}{\partial \phi}\right)^2 - \frac{m^2 g}{4}(\xi^2-\eta^2)$$

$$\Rightarrow \left[\frac{mE}{2}(\xi+\eta) - \{\;\} + \frac{m^2 g}{4}(\xi^2-\eta^2)\right]\frac{4\xi\eta}{\xi+\eta} = \left(\frac{\partial W_\phi}{\partial \phi}\right)^2$$

$$\frac{\partial W_\phi}{\partial \phi} = \alpha_1$$

$$\left[\frac{mE}{2}(\xi+\eta)-\left\{\eta\left(\frac{\partial W_\eta}{\partial\eta}\right)^2+\xi\left(\frac{\partial W_\xi}{\partial\xi}\right)^2+2m^2K\right\}+\frac{m^2g}{4}(\xi^2-\eta^2)\right]\frac{4\xi\eta}{\xi+\eta}$$

$$=\alpha_1^2$$

$$\Rightarrow \quad \frac{mE}{2}\xi-\xi\left(\frac{\partial W_\xi}{\partial\xi}\right)^2+2m^2K+\frac{m^2g}{4}\xi^2-\frac{\alpha_1^2}{4\xi}$$

$$=\frac{\alpha_1^2}{4\eta}-\frac{mE}{2}\eta+\eta\left(\frac{\partial W_\eta}{\partial\eta}\right)^2$$

* both sides are independent of eachother and must be equal to a const

$$\rightsquigarrow \quad \frac{\alpha_1^2}{4\eta}-\frac{mE}{2}\eta+\eta\left(\frac{\partial W_\eta}{\partial\eta}\right)^2=\alpha_2$$

$$\rightsquigarrow \quad \frac{mE}{2}\xi-\xi\left(\frac{\partial W_\xi}{\partial\xi}\right)^2+2m^2K+\frac{m^2g}{4}\xi^2-\frac{\alpha_1^2}{4\xi}=\alpha_2$$

d) $W_\phi = \alpha_1 \phi$

e)

$$\rightsquigarrow \frac{mE}{2} - \left(\frac{\partial W_\rho}{\partial \rho}\right)^2 + \frac{2m^2 k}{\rho} + \frac{m^2 g}{4}\rho - \frac{\alpha_1^2}{4\rho^2} - \frac{\alpha_2}{\rho} = 0$$

$$\Rightarrow W_\rho = \int \left(\frac{mE}{2} + \frac{2m^2 k}{\rho} + \frac{m^2 g}{4}\rho - \frac{\alpha_1^2}{4\rho^2} - \frac{\alpha_2}{\rho}\right)^{1/2} d\rho$$

$$\rightsquigarrow W_\eta = \int \left(\frac{\alpha_2}{\eta} + \frac{mE}{2} - \frac{\alpha_1^2}{4\eta^2}\right)^{1/2} d\eta,$$

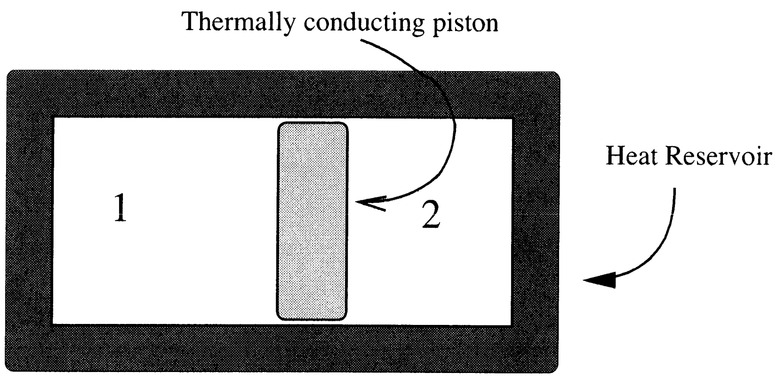
Statistical Mechanics

4. **Helmholtz Free Energy**: The Helmholtz free energy of an ideal monoatomic gas can be written as

$$F(T, V, N) = NkT \left\{ A - \log \left[T^{3/2} \frac{V}{N} \right] \right\}$$

where N is the total number of gas atoms, V is the volume, T is temperature, k is Boltzmann's constant and A is a dimensionless constant.

Consider a piston separating a system into two parts, with equal numbers of particles on the left and the right hand side. The whole system is in good thermal contact with a reservoir at constant temperature T. Initially, $V_1 = 2V_2$. The total volume, $V_{\text{tot}} = V_1 + V_2$, is fixed for this whole problem.



(a) Calculate the equilibrium position of the piston, once it is released. You must prove your answer, and not simply assert it. (3 points)

(b) Calculate the maximum available work the system can perform as it changes from the initial condition to the equilibrium position. (3 points)

(c) Calculate the change in the internal energy, U of gas 1 and gas 2 in the process. (2 points)

(d) Given your answers above, explain the source of energy for the work done during the expansion. (2 points)

4.

The Helmholtz free energy of an ideal monoatomic gas

$$F(T, V, N) = NKT \left\{ A - \log\left[T^{3/2} \frac{V}{N}\right] \right\}$$

(a) $F = E - TS$

$$\Rightarrow dF = dE - Tds - sdT$$

$$= \cancel{T}ds - PdV + \mu dN - T\cancel{ds} - SdT$$

$$= -PdV + \mu dN - SdT$$

$$\Rightarrow \left(\frac{\partial F}{\partial V}\right)_{N,T} = -P$$

* the system is in thermal equilibrium with the reservoir at temp T

* The system will be in equilibrium if $P_1 = P_2$

$$-P_1 = -NKT \frac{N}{V_1} T^{-3/2} \frac{T^{3/2}}{N} = -\frac{NKT}{V_1}$$

$$\Rightarrow P_1 = \frac{NKT}{V_1}$$ $\Rightarrow$ which is what we expect for an ideal gas

Similarly $P_2 = \frac{NKT}{V_2}$

Thus, at equilibrium, $P_1 = P_2$

$$\Rightarrow V_1 = V_2$$

* Hence the equilibrium pos should be at the middle

(b) * The expansion of V_2 & the compression of V_1 happens isothermally,

$$W_1 = +\int P\,dV$$

$$= +\int_{V_i}^{V_f} \frac{NKT}{V_1}\,dV$$

$$= NKT \ln\left(\frac{V_{1f}}{V_{1i}}\right)$$

$$= NKT \ln\left(\frac{3}{4}\right)$$

$V_{tot} = V_1 + V_2$

i

$V_{tot} = 3V_2$

$V_2 = \frac{V_{tot}}{3}$

$V_1 = \frac{2}{3} V_{tot}$

f

$V_{tot} = 2V_2$

$V_2 = V_1 = \frac{1}{2} V_{tot}$

$$W_2 = \int P\,dV = NKT \ln\left(\frac{V_{2f}}{V_{2i}}\right)$$

$$= NKT \ln\left(\frac{3}{2}\right)$$

$$W_{tot} = NKT \ln\left(\frac{9}{8}\right)$$

(c) $\Delta U = N\frac{f}{2}k\Delta T$ but it is an isothermal process

& $\Delta T = 0$

$\therefore \Delta U_1 = 0$

$\Delta U_2 = 0$

$Q_1 = NkT\ln\left(\frac{3}{4}\right)$

d) $\Delta U = Q - W$

for $\Delta U = 0$ $Q_2 = NkT\ln\left(\frac{3}{2}\right)$

$\Rightarrow Q = W$

So, the source of work done is the transfer

from the reservoir

5. Consider a gas of N non-interacting **one dimensional** diatomic molecules enclosed in a box of "volume" L (actually, just a length) at temperature T.

   (a) The classical energy for a single molecule is:

   $$E(p_1, p_2, x_1, x_2) = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{1}{2}K(x_1 - x_2)^2$$

   where p_1 and p_2 are the classical momenta of the atoms in one diatomic molecule, x_1 and x_2 are their classical positions, and K is the spring constant. Calculate the specific heat for the gas. (You should assume that $KL^2/2 >> k_B T$, where k_B is Boltzmann's constant.) (4 points).

   (b) In the quantum limit the energy levels of the molecule are discrete. In a semiclassical approach we can write the energy of one molecule as:

   $$E(P, n) = \frac{P^2}{4m} + \hbar\omega(n + \frac{1}{2})$$

   where P is the momentum of the diatomic molecule (of mass $2m$), and ω is the natural frequency of the oscillator, and n is a non-negative integer ($n \geq 0$). Calculate the specific heat. (4 points).

   (c) Calculate the high and low temperature limits of your result in (b), and explain how they relate to the result of (a). (2 points)

5.

a) $$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{1}{2}K(x_1-x_2)^2$$

* Notice: the system has 3 degrees of freedom.
Each atom can move in $\underbrace{\text{one direction}}_{1\times 2}$ and they can $\underbrace{\text{vibrate}}_{1}$.

Make a co-ordinate transformation such that,

$$X_{cm} = \frac{X_1+X_2}{2} \quad \text{and} \quad x = X_1 - X_2$$

$$L = \frac{1}{2}(m+m)\dot{X}_{cm}^2 + \frac{1}{2}\left(\frac{mm}{m_1+m_2}\right)\dot{x}^2 - \frac{1}{2}Kx^2$$

$$= \frac{m}{4}(\dot{x}_1^2 + \dot{x}_2^2 + \cancel{2\dot{x}_1\dot{x}_2}) + \frac{m}{4}(\dot{x}_1^2 + \dot{x}_2^2 - \cancel{2\dot{x}_1\dot{x}_2}) - \frac{1}{2}K(x_1-x_2)^2$$

$$= \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) + \frac{1}{2}K(x_1-x_2)^2 \Rightarrow$$ which is the original lagrangian

Thus, the new Hamiltonian becomes

$$H = \frac{P_{cm}^2}{4m} + \frac{P_x^2}{m} + \frac{1}{2}Kx^2$$

$$Z_1 = \frac{1}{2\pi\hbar}\int_0^L\int_{-\infty}^{+\infty} dX_{cm}\, dP_{cm}\, e^{-\beta\frac{P_{cm}^2}{4m}} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dx\, dp_x\, e^{-\frac{\beta P_x^2}{m}} e^{-\frac{\beta K}{2}x^2}$$

as, $\frac{1}{2}KL^2 \gg K_BT$, we can integrate $x \equiv (x_1-x_2)$ from $-\infty$ to $+\infty$

$$Z_1 = \frac{L}{2\pi\hbar} \sqrt{4\pi m k_B T} \sqrt{\pi m k_B T} \sqrt{\frac{2\pi k_B T}{k}}$$

$$= \alpha \cdot \beta^{-3/2}$$

$\uparrow$

rest of the stuff

$$Z = \frac{1}{N!} Z_1^N = \frac{(\alpha \beta^{-3/2})^N}{N!}$$

$$\Rightarrow \quad E = -\frac{\partial \ln Z}{\partial \beta} = +\frac{3N}{2} \frac{1}{\beta}$$

$$\rightsquigarrow c_V = \frac{\partial E}{\partial T} = -k\beta^2 \frac{\partial E}{\partial \beta} = \frac{3Nk}{2}$$

b) $$E(p,n) = \underbrace{\frac{p^2}{4m}}_{H_0} + \underbrace{\hbar\omega\left(n+\tfrac{1}{2}\right)}_{H'}$$

$$Z_1^0 = \frac{L}{2\pi\hbar}\sqrt{4\pi m k_B T}$$

$$Z_1' = \sum_{n=0}^{\infty} e^{-\beta\hbar\omega(n+\frac{1}{2})} = e^{-\frac{\beta\hbar\omega}{2}}\sum_{n=0}^{\infty}\left(e^{-\beta\hbar\omega}\right)^n$$

$$= \frac{e^{-\beta\hbar\omega/2}}{1-e^{-\beta\hbar\omega}}$$

$$Z_1 = Z_1^0 Z_1' = \underbrace{L\sqrt{\frac{m}{\pi\hbar^2}}}_{\alpha}\,\beta^{-1/2}\,\frac{e^{-\beta\hbar\omega/2}}{1-e^{-\beta\hbar\omega}}$$

$$Z^N = Z_1^N = \left(\alpha\beta^{-1/2}\frac{1}{e^{\beta\hbar\omega/2}-e^{-\beta\hbar\omega/2}}\right)^N$$

$$E = -\frac{\partial \ln Z}{\partial\beta} = -\frac{\partial}{\partial\beta}\left\{\ln\beta^{-N/2} + \ln\left(e^{\beta\hbar\omega/2}-e^{-\beta\hbar\omega/2}\right)^{-N} + \ln\alpha\right\}$$

$$= \frac{N}{2}\frac{1}{\beta} + N\,\frac{\frac{\hbar\omega}{2}\left(e^{\beta\hbar\omega/2}+e^{-\beta\hbar\omega/2}\right)}{\left(e^{\beta\hbar\omega/2}-e^{-\beta\hbar\omega/2}\right)}$$

$$= \frac{N}{2}\frac{1}{\beta} + \frac{N\hbar\omega}{2}\coth\left(\frac{\beta\hbar\omega}{2}\right)$$

$$C_V = -k\beta^2\frac{\partial E}{\partial\beta} = \frac{N}{2} + \frac{N\hbar^2\omega^2}{4}k\beta^2\,\mathrm{csch}^2\left(\frac{\beta\hbar\omega}{2}\right)$$

c) for $T \to \infty$ $\quad \frac{\hbar\omega}{2KT} << 1$

$$\operatorname{cosch} x \approx \frac{1}{x} = \frac{2KT}{\hbar\omega}$$

$$C_v \approx \frac{NK}{2} + \frac{N\hbar^2\omega^2}{4K}\beta^2 \frac{4}{\hbar^2\omega^2\beta^2}$$

$$\approx \frac{NK}{2} + NK$$

$$= \frac{3NK}{2}$$

* As expected in high temp limit the energy levels are close to each other comped to the total "volume" that it can be treated as classical diatomic gas

6. Fermions:

   (a) Show that for any non-interacting spin 1/2 fermionic system with chemical potential μ, the probability of occupying a single particle state with energy $\mu + \delta$ is the same as finding a state vacant at an energy $\mu - \delta$. (2 points)

   (b) Consider non-interacting fermions that come in two types of energy states:

$$E_{\pm}(\vec{k}) = \pm\sqrt{m^2c^4 + \hbar^2 k^2 c^2}$$

   At zero temperature all the states with negative energy (all states with energy $E_{-}(\vec{k})$) are occupied[1] and all positive energy states are empty, and that $\mu(T = 0) = 0$. Show that the result of part (a) above means that the chemical potential must remain at zero for all temperatures if particle number is to be conserved. (2 points)

   (c) Using the results of (a) and (b) above, show that the average excitation energy, the change in the energy of the system from it's energy at $T = 0$ in three dimensions is given by:

$$\Delta E \equiv E(T) - E(0) = 4V \int \frac{d\vec{k}}{(2\pi)^3} E_{+}(\vec{k}) \frac{1}{1 + e^{\beta E_{+}(\vec{k})}}$$

   (2 points)

   (d) Evaluate the integral above for massless ($m = 0$) particles. (2 points)

   (e) Calculate the heat capacity of such particles. (2 points)

[1] Technically this means the total energy of the system diverges. If this bothers you, you can assume some large cut-off to the wavevectors, $\hbar k_{\mathrm{max}} c >> kT$, which will have no effect on your final answers.

6.

a) * Probability of finding a single particle state with energy $\mu+\delta$ is

$$P(\mu+\delta) = f(\mu+\delta) = \frac{1}{e^{\beta(E-\mu)}+1} = \frac{1}{e^{\beta\delta}+1}$$

* Probability of finding the state $\mu-\delta$ vacant is

$$P'(\mu-\delta) = 1 - \frac{1}{e^{-\beta\delta}+1} = \frac{e^{-\beta\delta}}{e^{-\beta\delta}+1} = \frac{1}{e^{\beta\delta}+1}$$

$$\therefore \; P(\mu+\delta) = P'(\mu-\delta)$$

b) At, $T=0$, $\mu=0$

$$\therefore \; P(\delta) = P'(-\delta)$$

* So if N is the the no of particles in E_- state then in order to conserve particle no this is the same amount of particles NOT present in the state E_+

where, $E_- = -E_+$

$$N\,P(E_-) = N(1-P(E_+))$$

$$\Rightarrow \frac{N}{e^{\beta(E_- -\mu)}+1} = N\left(1 - \frac{1}{e^{\beta(E_+ -\mu)}+1}\right)$$

$$\Rightarrow \quad \frac{1}{e^{-\beta(E_+ + \mu)} + 1} = \frac{e^{\beta(E_+ - \mu)}}{e^{\beta(E_+ - \mu)} + 1}$$

$$\Rightarrow \quad 1 + e^{-\beta(E_+ + \mu)} = 1 + e^{-\beta(E_+ - \mu)}$$

$$\Rightarrow \quad E_+ + \mu = E_+ - \mu$$

$$\Rightarrow \quad \mu = 0$$

Hence, $\mu = 0$ for any T

c) The average energy is

$$\langle E \rangle = \frac{\overset{g_s}{\downarrow}}{} \frac{2V}{(2\pi)^3} \int_{-\infty}^{+\infty} d^3K \; f(E) \, dE$$

$$= \frac{2V}{(2\pi)^3} \int_{-\infty}^{\infty} d\vec{K} \; f(E)$$

$$f(E_+)\big|_{T=0} = \frac{1}{e^{+\infty} + 1} \approx 0$$

$$f(E_-)\big|_{T=0} = \frac{1}{e^{-\infty} + 1} \approx 1$$

$$E(T) = \frac{2V}{(2\pi)^3} \int d^3K \left[E_+ P(E_+) + \underbrace{E_- P(E_-)}_{\substack{\shortparallel \\ -E_+ (1 - P(E_+))}} \right]$$

$$= \frac{4V}{(2\pi)^3} \int d^3K \, E_+ P(E_+) - \frac{2V}{(2\pi)^3} \int d^3K \, E_+$$

$$E(0) = \frac{2V}{(2\pi)^3} \int d^3K \, E_-$$

$$= - \frac{2V}{(2\pi)^3} \int d^3K \, E_+$$

$$\Delta E \equiv E(T) - E(0)$$

$$= 4V \int \left(\frac{dk}{2\pi} \right)^3 E_+(\bar{K}) \frac{1}{1 + e^{\beta E_+(\bar{K})}}$$

d) For $m = 0$

$$E_+ = \hbar k c \qquad dE_+ = \hbar c\, dk$$

$$k = \frac{E_+}{\hbar c}$$

$$\Delta E = \frac{4V}{(2\pi)^3} 4\pi \int k^2 dk \frac{E_+(k)}{e^{\beta E_+} + 1}$$

$$= \frac{4V}{(2\pi)^3} 4\pi \frac{1}{(\hbar c)^4} \int_0^\infty \frac{E_+}{e^{\beta E_+} + 1} dE_+$$

$$\beta E_+ = x$$

$$dE_+ = \frac{1}{\beta} dx$$

$$E_+^3 = \frac{x^3}{\beta^3}$$

$$= \frac{16\pi V}{8\pi^3} \frac{1}{(\beta \hbar c)^4} \underbrace{\int_0^\infty \frac{x^3}{e^x + 1} dx}_{\Gamma(4)\, f_4(1) = 6\zeta(-4)}$$

$$= 0$$

$$(1 - 2^{-s})\, s!\, \zeta(s+1)$$