Classical Mechanics and Statistical/Thermodynamics

August 2007

Possibly Useful Information

Handy Integrals:

$$\int_0^\infty x^n e^{-\alpha x}\,dx = \frac{n!}{\alpha^{n+1}}$$

$$\int_0^\infty e^{-\alpha x^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$$

$$\int_0^\infty x\,e^{-\alpha x^2}\,dx = \frac{1}{2\alpha}$$

$$\int_0^\infty x^2\,e^{-\alpha x^2}\,dx = \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}}$$

$$\int_{-\infty}^\infty e^{i\,a\,x-b\,x^2}\,dx = \sqrt{\frac{\pi}{b}}e^{-a^2/4b}$$

Geometric Series:

$$\sum_{n=0}^\infty x^n = \frac{1}{1-x} \qquad \text{for} \quad |x|<1$$

Stirling's approximation:

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

Riemann and related functions:

$$\sum_{n=1}^\infty \frac{z^n}{n^p} \equiv Li_p(z) = \begin{cases} g_p(z) & \text{if } z \geq 0 \\ f_p(z) & \text{if } z < 0 \end{cases}$$

$$\sum_{n=1}^\infty \frac{1}{n^p} \equiv \zeta(p)$$

The function $Li_p(z)$ is the polylog function, and it is sometimes referred to in statistical mechanics using the g and f functions as noted above.

$$g_p(1) = \zeta(p) \qquad\qquad f_p(1) = \zeta(-p)$$

$$\begin{array}{rr} \zeta(1) = \infty & \zeta(-1) = 0.0833333 \\ \zeta(2) = 1.64493 & \zeta(-2) = 0 \\ \zeta(3) = 1.20206 & \zeta(-3) = 0.0083333 \\ \zeta(4) = 1.08232 & \zeta(-4) = 0 \end{array}$$

F-07

Classical Mechanics

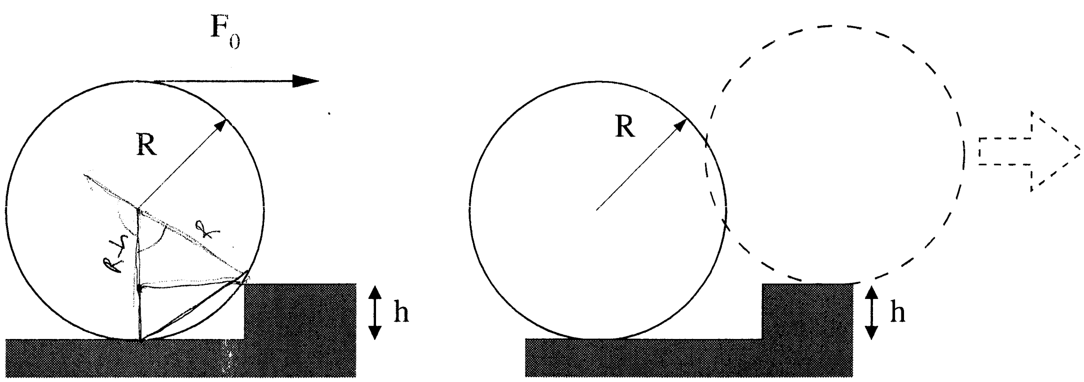
1. A uniform sphere of radius R and mass m encounters a step of height h, where $h < R$. At all times it rolls without slipping.

   (a) Consider the case on the left where the sphere is at rest, touching the edge of the step (figure (a)).

      i. Calculate the magnitude of the force F_0 that must be applied in the horizontal direction, tangent to the sphere, in order to push the sphere over the step. **(2 points)**

      ii. Calculate the magnitude of the force on the sphere from the edge of the step at the point of contact. **(2 points)**

   (b) Consider the case where the sphere starts with speed v_0 and you do not apply an external force. What is the minimum value of v_0 that will allow the ball to mount the step, and continue rolling along the surface, as shown in figure (b)? Assume that there is no slipping at the point of contact between the sphere and the step. (Note: this is not a trivial conservation of energy problem, since the collision with the step is *inelastic*.) **(6 points)**

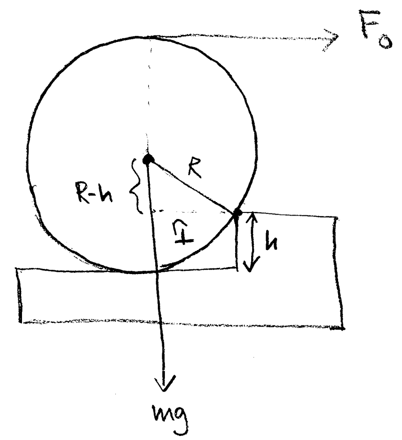


(a) (b)

1.

a)

i. to just push the sphere over the step, the net torque for all the forces about the pivot pt. should be zero



$$\tau_{net} = \tau_{F_g} + \tau_{F_o} + \tau_{F_{\text{edge of the step}}} = 0$$

$$\tau_{F_o} = \bar{r} \times \bar{F_o} = \underset{\uparrow \atop CW}{-} \underbrace{r \sin\theta}_{= r_\perp} F_o = (2R-h)F_o$$

$$\tau_{F_g} = \underset{\uparrow \atop CCW}{+} r_\perp mg = m\sqrt{R^2-(R-h)^2}$$

$$\tau_{F_{edge}} = \underbrace{r_\perp}_{=0} F_{edge} = 0$$

$$\tau_{net} = m\sqrt{R^2+(R-h)^2} - (2R-h)F_o = 0$$

$$F_o = \frac{m\left(R^2-(R-h)^2\right)^{1/2}}{(2R-h)} = \frac{m\left(2Rh-h^2\right)^{1/2}}{(2R-h)}$$

ii the net force on the sphere has to be zero

$$\rightsquigarrow \sum F_x = 0$$

$$\Rightarrow F_0 - (F_{edge})_x = 0$$

$$\Rightarrow (F_{edge})_x = F_0$$

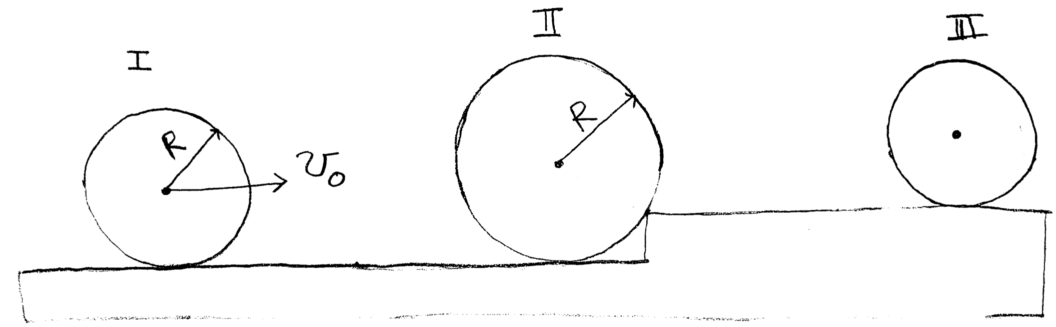
$$\rightsquigarrow \sum F_y = 0$$

$$\Rightarrow (F_{edge})_y - mg = 0$$

$$\Rightarrow (F_{edge})_y = mg$$

$$\bar{F}_{edge} = \begin{pmatrix} F_0 \\ mg \end{pmatrix}$$

(b)



* **At I**, the velocity is v_0

so, angular momentum $L_0 = I_0 \omega_0$

* For a sphere about com

$I_0 = \frac{2}{5} mR^2$

* **At II**, when the ball touches the edge of the step, the pivot pt. changes,

So, the moment of inertia changes to,

$$I_1 = I_0 + mR^2 = \frac{7}{5} mR^2$$

→ the collision of the sphere with the edge is inelastic, so the angular momentum is conserved, thus

* Using conservation of linear momentum

$m v_0 = m v_0$

$$L_0 = L_1$$

$$\Rightarrow I_0 \omega_0 = I_1 \omega_1$$

$$\Rightarrow \omega_1 = \frac{I_0 \omega_0}{I_1} = \frac{(2/5\, mR^2)(v_0)}{7/5\, mR^2\, R} = \frac{2}{7} \frac{v_0}{R}$$

* Right after the collision the total energy is

$$E_{II} = \frac{1}{2} m v_0^2 + \frac{1}{2} I_1 \omega_1^2$$

At III, $E_{III} = mgh$

Use conservation of energy

$$\frac{1}{2} m v_0^2 + \frac{1}{2} I_1 \omega_1^2 = mgh$$

$$v_0^2 = 2gh - \frac{I_1}{m} \omega_1^2$$

$$= 2gh - \frac{7}{5} R^2 \left(\frac{2}{7} \frac{v_0}{R}\right)^2$$

$$v_0^2 = 2gh - \frac{4 v_0^2}{35}$$

$$\Rightarrow v_0^2 \left(1 + \frac{4}{35}\right) = 2gh$$

$$\Rightarrow v_0 = \sqrt{\frac{70}{39} gh}$$

2. A light, perfectly elastic ball (e.g. a ping-pong ball) is dropped from a large height on to a flat, horizontal surface. The acceleration due to gravity can be taken as a constant, g. The ball reaches its terminal velocity, v_0 and then rebounds elastically from the ground at time $t = 0$. The effect of air friction can be modelled by a velocity dependent damping force:

$$F_{\text{fric}} = -\beta\, v(t) = -\beta\, \dot{y}(t).$$

(a) Derive an expresion for v_0 as a function of β, m, and g. **(1 points)**

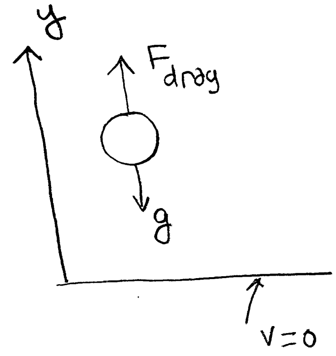
(b) Calculate $v(t)$ for $0 < t < t_b$, where t_b is the time of the next bounce. **(5 points)**

(c) Given that the ball rebounds to a maximum height h, calculate a value for β from m, h, and g. **(4 points)**

2.

a) $$L = \frac{1}{2} m \dot{y}^2 - mgy$$

$$v_0 = \frac{mg}{\beta}$$



$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = F_{drag}$$

$$\Rightarrow m\ddot{y} + mg = -\beta \dot{y}(t)$$

$$\Rightarrow \ddot{y} + \frac{\beta}{m}\dot{y} = -g$$ $\Rightarrow$ For drag force $\ddot{y} = 0 \Rightarrow \dot{y} = \frac{mg}{\beta}$

$$\Rightarrow v_0 = -\frac{mg}{\beta}$$

b) $$\Rightarrow Y_c(t) = A e^{-\frac{\beta}{m}t}$$

$$Y_p = Ct^2 + Dt + E$$

$$\dot{Y} = 2Ct + D$$

$$\ddot{Y} = 2C$$

$$2C + \frac{\beta}{m}(2Ct + D) = -g$$

$$\Rightarrow \quad C = 0$$
$$D = \frac{mg}{\beta}$$
$$E = 0$$

$$Y_p(t) = -\frac{mg}{\beta}t$$

So, $$Y(t) = Ae^{-\frac{\beta}{m}t} - \frac{mg}{\beta}t$$

$$\dot{Y}(t) = -A\sqrt{\frac{\beta}{m}}\,\sin\left(\sqrt{\frac{\beta}{m}}\right)t + B\sqrt{\frac{\beta}{m}}\,\cos\left(\sqrt{\frac{\beta}{m}}\right)t + \frac{mg}{\beta}$$

At $t=0$ $\dot{Y}(0) = v_0 = \frac{mg}{\beta}$

$$\sqrt{\frac{\beta}{m}}\,B = -\frac{mg}{\beta} \Rightarrow B = -\left(\frac{m}{\beta}\right)^{3/2} g$$

$$\dot{Y}(t) = -A\sqrt{\frac{\beta}{m}}\,\sin\left(\sqrt{\frac{\beta}{m}}\right)t - \frac{m}{\beta}\cos\left(\sqrt{\frac{\beta}{m}}\right)t + \frac{mg}{\beta}$$

c) At $Y=h$, $\dot{Y}(h) = 0$

$$\dot{Y}_f^2 = Y_i^2 + 2gh$$

$$\dot{y}(t) = -\frac{\beta}{m} A e^{-\frac{\beta}{m}t} - \frac{mg}{\beta}$$

At, $t = 0$, $\dot{y}(0) = +\frac{mg}{\beta}$ * as after rebounding the ball moves upward so sign is +

$$\Rightarrow \frac{2mg}{\beta} = -\frac{\beta}{m} A$$

$$\Rightarrow A = -\frac{2m^2 g}{\beta^2}$$

$$\dot{y}(t) = \frac{2mg}{\beta} A e^{-\frac{\beta}{m}t} - \frac{mg}{\beta}$$

$$\dot{y}(t) = \frac{mg}{\beta}\left(2 e^{-\beta/m t} - 1\right)$$

c) $$Y(t) = -\frac{mg}{\beta}\left[\frac{2m}{\beta} e^{-\frac{\beta}{m}t} + t\right] + C$$

At, $t = 0$ $Y(0) = 0$

$$C = \frac{2m^2 g}{\beta^2}$$

$$Y(t) = \frac{2m^2 g}{\beta^2}\left(1 - e^{-\frac{\beta}{m}t}\right) - \frac{mg}{\beta} t$$

At, h, $\dot{Y} = 0$

$$\alpha e^{-\beta/m t} - 1 = 0$$

$$\Rightarrow \quad -\frac{\beta}{m} t = -\ln 2$$

$$\Rightarrow \quad t = \frac{m \ln 2}{\beta}$$

Then

$$h = \frac{2m^2 g}{\beta^2} \underbrace{\left(1 - e^{-\frac{\beta}{m}\frac{m}{\beta}\ln 2}\right)}_{= \frac{1}{2}} - \frac{mg}{\beta}\frac{m}{\beta}\ln 2$$

$$= \frac{m^2 g}{\beta^2} - \frac{m^2 g}{\beta^2}\ln 2 = \frac{m^2 g}{\beta^2}(1 - \ln 2)$$

$$\Rightarrow \quad \beta = m\sqrt{\frac{g}{h}(1 - \ln 2)}$$

3. Consider the Lagrangian for the harmonic oscillator with generalized coordinate q:

$$L = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2$$

(a) Explicitly derive the Hamiltonian from the Lagrangian. Do not simply state a result. **(1 point)**

(b) Write the Hamilton-Jacobi differential equation for the system. **(2 points)**

(c) Use the separation ansatz for the action, $S = S_1(t) + S_2(q)$ to obtain differential equations for S_1 and S_2. **(2 points)**

(d) Solve your equations for S_1 and S_2. **(2 points)**

(e) Invert your results to find an expression for $q(t)$, using $\omega \equiv \sqrt{k/m}$. **(2 points)**

(f) Are your results consistent with your knowledge of the harmonic oscillator? Discuss. **(1 point)**

3.

$$L = \frac{1}{2} m \dot{q}^2 - \frac{1}{2} k q^2$$

a) $$p = \frac{\partial L}{\partial \dot{q}} = m\dot{q}$$

$$H = \frac{1}{2} m \dot{q}^2 + \frac{1}{2} k q^2 = \frac{p^2}{2m} + \frac{1}{2} k q^2$$

b) To write the Hamilton-Jacobi differential eqn we make a Legendre transformation of H such that, the new Hamiltonian is a const of motion

$$K = H + \frac{\partial S(q, \alpha, t)}{\partial t}$$

$S(q, \alpha, t)$ is the generating func / Action / Hamilton's principle func

without loss of generality, $K = 0$

$$H(q, p) + \frac{\partial S(q, \alpha, t)}{\partial t} = 0$$

Now, $$p = \frac{\partial S}{\partial q}$$

$$H\left(q, \frac{\partial S}{\partial q}\right) + \frac{\partial S}{\partial t} = 0$$

Thus,

$$\frac{1}{2m}\left(\frac{\partial S}{\partial t}\right)^2 + \frac{1}{2} k q^2 + \frac{\partial S}{\partial t} = 0$$

c) $S(q,t) = S_1(q) + S_2(t)$

* Notice Hamiltonian is time <u>independent</u>

$$H\left(q, \frac{\partial S}{\partial q}\right) \equiv H\left(q, \frac{\partial S_1(q)}{\partial q}\right)$$

So,

$$\frac{1}{2m}\left(\frac{\partial S_1(q)}{\partial q}\right)^2 + \frac{1}{2}kq^2 = -\frac{\partial S_2(t)}{\partial t}$$

* both sides are independent of each other

so they must be equal to a const.

$$\rightsquigarrow \quad \frac{\partial S_2(t)}{\partial t} = -E$$

$$\rightsquigarrow \quad \frac{1}{2m}\left(\frac{\partial S_1(q)}{\partial q}\right)^2 + \frac{1}{2}kq^2 = E$$

d) $$\frac{\partial S_2(t)}{\partial t} = -E$$

$$\Rightarrow \quad S_2(t) = -Et$$

$\uparrow$

* This solution shows

$H = E$

$$\frac{\partial S_1(q)}{\partial q} = \sqrt{2mE - mkq^2}$$

$$S_1(q) = \int \sqrt{2mE - mkq^2}$$

* to solve for $q(t)$ it would be easier to evaluate, $Q = \frac{\partial S_1(q)}{\partial P}$ and then solve the integral

e) $$Q = \frac{\partial}{\partial P} \int \sqrt{2mE - mkq^2}\, dq$$

$$= \frac{\partial}{\partial E} \int \sqrt{2mE - mkq^2}\, dq$$

$$= \int \frac{1}{2} \frac{2m}{\sqrt{2mE - m^2\omega^2 q^2}}\, dq$$

$$= \int \frac{m}{m\omega \sqrt{\frac{2E}{m\omega^2} - q^2}}\, dq$$

$$q = \sqrt{\frac{2E}{m\omega^2}} \sin\theta$$

$$dq = \sqrt{\frac{2E}{m\omega^2}} \cos\theta\, d\theta$$

$$= \sqrt{\frac{2E}{m\omega^2}} \frac{1}{\omega} \sqrt{\frac{2E}{m\omega^2}}\ \theta$$

$$= \frac{1}{\omega} \sin^{-1}\left(\sqrt{\frac{m\omega^2}{2E}}\, q\right)$$

$$q = \sqrt{\frac{2E}{m\omega^2}} \sin(\omega Q)$$

$$\rightsquigarrow \dot{Q} = \frac{\partial H}{\partial P} = \frac{\partial E}{\partial E} = 1$$

$$Q(t) = t + \text{const}$$

$$q(t) = \sqrt{\frac{2E}{m\omega^2}} \sin(\omega(t - t_0))$$

f) $$x(t) = A \sin\omega(t - t_0)$$

$$A = \sqrt{\frac{2E}{m\omega^2}}$$

Statistical Mechanics

4. Consider a certain hard sphere model of a gas of N particles in which we have an "excluded volume term." The entropy in this case can be given as:

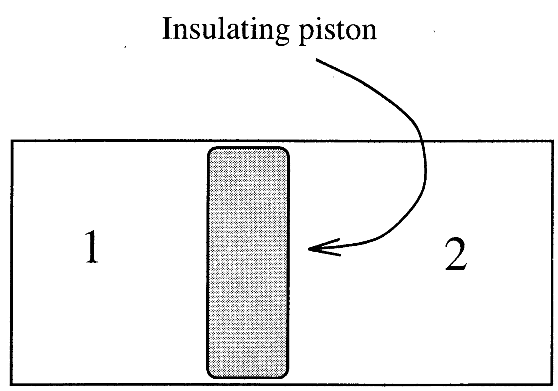
$$S(U, V, N) = Nk \ln \left[(V - Nb) \frac{U}{\epsilon_0 N^2} \right]$$

Here b represents the volume of one gas particle, V is the volume of the container, U is the internal energy of the gas, and k is Boltzmann's constant. The constant ϵ_0 has dimensions of energy $\times$ volume, and is included to keep the argument of the logarithm dimensionless.

   (a) Does this system satisfy the third law of thermodynamics (i.e. does the entropy of the system go to zero as the temperature goes to zero)? Prove your answer. **(3 points)**

   (b) What is the specific heat at constant volume for this gas? **(2 points)**

   (c) A gas with N_1 particles with total energy U_1 in a volume V_1 has an excluded volume/particle of b_1. It is separated by a moveable, insulating piston from a second gas of N_2 particles with total energy U_2 in a volume V_2 and an excluded volume/particle of b_2. The piston is allowed to move so that $V_{\text{tot}} = V_1 + V_2$ is a constant, but V_1 and V_2 can change. What is the value of V_1 in equilibrium? **(5 points)**



4.

$$S(E, V, N) = NK \ln\left[(V-Nb)\frac{E}{\epsilon_0 N^2}\right]$$

a) $dE = T\,dS - P\,dV + \mu\,dN$

$$\left.\frac{\partial S}{\partial E}\right|_{V,N} = \frac{1}{T}$$ * Consider V & N of the gas is not changing

$$\Rightarrow \frac{1}{T} = \frac{NK}{(V-Nb)\frac{E}{\epsilon_0 N^2}} \frac{(V-Nb)}{\epsilon_0 N^2} = \frac{NK}{E}$$

$$\Rightarrow E = NKT$$

As, $T \to 0$ $\quad S \to NK \ln(0) \to -\infty$

* Shows entropy goes to a minimum. In the low temp limit QM effects are significant so this eqn doesn't hold at that limit. But still as an appx we can say the entropy goes to zero and 3rd law is satisfied.

$\rightsquigarrow$ It might also be safe to argue

Since, $S = K \ln \Omega(N, V, E)$

As $T \to 0$ the no of mictrostate is 1

$S = \ln 1 = 0$

b) $C_V = \frac{\partial E}{\partial T} = \frac{Nk}{V-nb}$

c) $dE = Tds - pdV + \mu dN$

$$\left.\frac{\partial s}{\partial E}\right|_{V,N} = \frac{1}{T} \quad ; \quad \left.\frac{\partial s}{\partial V}\right|_{E,N} = \frac{P}{T} \quad ; \quad \left.\frac{\partial s}{\partial N}\right|_{E,V} = -\frac{\mu}{T}$$

$$P = \frac{(\partial s/\partial V)_{E,N}}{(\partial s/\partial E)_{V,N}} = -\left(\frac{\partial E}{\partial V}\right)_{N,S}$$

$$S = NK \ln\left[(V-Nb)\frac{E}{\epsilon_0 N^2}\right]$$

$$\left.\frac{\partial S_1}{\partial E_1}\right|_{V_1,N_1} = N_1 K T_1$$

$$\left.\frac{\partial S_1}{\partial V_1}\right|_{E_1,N_1} = \frac{N_1 K}{(V_1 - N_1 b_1)\frac{E_1}{\epsilon_0 N_1^2}} \times \frac{E_1}{\epsilon_0 N_1^2} = \frac{N_1 K}{V_1 - N_1 b_1}$$

$$\therefore \frac{P_1}{T_1} = \frac{N_1 K}{(V_1 - N_1 b_1)} \Rightarrow P_1 = \frac{N_1 K T_1}{V_1 - N_1 b_1}$$

$$\Rightarrow P_2 = \frac{N_2 K T_2}{V_2 - N_2 b_2}$$

At equilibrium,

$$P_1 = P_2$$

$$\frac{N_1 K T_1}{V_1 - N_1 b_1} = \frac{N_2 K T_2}{V_2 - N_2 b_2}$$

$$\Rightarrow \quad V_1 = \frac{N_1 K T_1}{N_2 K T_2}\left(V_2 - N_2 b_2\right) + N_1 b_1$$

5. A crystalline solid contains N similar, immobile, statistically independent defects. Each defect has 5 possible states $s_1, s_2, \ldots s_5$. The energies of the states are given by $E_1 = E_2 = 0$, and $E_3 = E_4 = E_5 = \Delta$.

   (a) Find the partition function for the defects as a function of their number, and the temperature T. **(3 points)**

   (b) Find the defect contribution to the entropy of the crystal as a function of Δ and the temperature T. **(4 points)**

   (c) Without doing a detailed calculation state the contribution to the internal energy due to the defects in the limit $kT >> \Delta$. Explain your reasoning. **(3 points)**

* entropy is max with equal probability

5.

a) one-defect partion func

$$Z_1 = \sum_{\epsilon_i} e^{-\beta \epsilon_i}$$

$$Z_1 = 2 + 3e^{-\beta\Delta}$$

total partition func,

$$Z = Z_1^N = (2+3e^{-\beta\Delta})^N$$

b) $$F = -KT \ln Z = -NKT\ln(2+3e^{-\beta\Delta})$$

$$S = -\left.\frac{\partial F}{\partial T}\right|_{V,N}$$

$$= NK\ln(2+3e^{-\beta\Delta}) + NKT\frac{(-3\Delta e^{-\beta\Delta})}{(2+3e^{-\beta\Delta})}$$

$$= NK\ln(2+3e^{-\beta\Delta}) - \frac{3N\Delta KT}{2+3e^{-\beta\Delta}}e^{-\beta\Delta}$$

c)

$$E = -\frac{\partial}{\partial\beta}\ln Z$$

$$= -N\ln\left(2+3e^{-\beta\Delta}\right)$$

$$= -\frac{N}{2+3e^{-\beta\Delta}}\left(-3\Delta e^{-\beta\Delta}\right)$$

$$= 3N\Delta\,\frac{e^{-\Delta/KT}}{2+3e^{-\Delta/KT}}$$

$$= \frac{3N\Delta}{2e^{\Delta/KT}+3}$$

$$= \frac{3N\Delta}{2\left(1-\frac{\Delta}{KT}\right)+3}$$

$$= \frac{3N\Delta}{5-\frac{2\Delta}{KT}}$$

$$= \frac{3N\Delta}{5}\left(1-\frac{2\Delta}{5KT}\right)^{-1}$$

$$= \frac{3N\Delta}{5}\left(1+\frac{2\Delta}{5KT}\right)$$

At $KT \gg \Delta$, $E = \frac{3N\Delta}{5}$

At, High temp the entropy of the system is max, and the entropy is max when all the states are equally probable to populate. So each state has $P(E)=\frac{1}{5}$ but $E=0$ for E_1 & E_2 Hence, $E = \frac{3N\Delta}{5}$

6. Consider a set ($N >> 1$) of spinless bosons confined in a harmonic oscillator potential. The characteristic frequency of the harmonic potential is ω_0, and $\hbar\omega_0 << kT$, where T is the temperature and k is Boltzmann's constant.

   (a) Assuming the system is one dimensional, so that the energy of the system is given by $E = \hbar\omega_0(n + 1/2)$, calculate $N(T, V, \mu)$, in the above limit, where μ is the chemical potential. **(3 points)**

   (b) Show that there is no Bose-Einstein transition for this system in 1D. **(1 points)**

   (c) Assuming the system is two dimensional, calculate $N(T, V, \mu)$, again in the limit $\hbar\omega_0 << kT$. **(3 points)**

   (d) Show that there is a Bose-Einstein transtion and calculate the critical temperature as a function of the number of particles. (Do **not** simply quote a result.) **(3 points)**

6.

The energy of one boson is

$$\epsilon_i = \hbar\omega_0 (i + \tfrac{1}{2})$$

$$Q = \sum_{N=0}^{\infty} e^{\beta\mu N} \sum_{\{n_i\}} e^{-\beta N \epsilon_i}$$

$$= \sum_{N=0}^{\infty} \sum_{\{n_i\}} e^{\sum_i \beta[\mu - \hbar\omega_0(i+\frac{1}{2})] n_i}$$

$$= \sum_{\{n_i\}=0}^{\infty} \prod_i e^{\beta[\mu - \hbar\omega_0(i+\frac{1}{2})] n_i}$$

$$= \prod_i \frac{1}{1 - e^{-\beta[\hbar\omega_0(i+\frac{1}{2}) - \mu]}}$$

$$\langle N \rangle = -\left.\frac{\partial \mathcal{G}}{\partial \mu}\right|_{V,T}$$

$$= -KT \frac{\partial \ln Q}{\partial \mu}$$

$$= \sum_i KT \frac{\partial}{\partial \mu} \ln\left(1 - e^{-\beta[\hbar\omega_0(i+\frac{1}{2}) - \mu]}\right)$$

$$\langle N \rangle = \sum_i KT \frac{\beta\, e^{-\beta[\hbar\omega_0(i+\frac{1}{2})-\mu]}}{1-e^{-\beta[\hbar\omega_0(i+\frac{1}{2})-\mu]}}$$

$$= \sum_i \frac{e^{-\beta[\hbar\omega_0(i+\frac{1}{2})-\mu]}}{1-e^{-\beta[\hbar\omega_0(i+\frac{1}{2})-\mu]}} = \sum_i \frac{1}{e^{\beta[\hbar\omega_0(i+\frac{1}{2})-\mu]}-1}$$

Since, $\hbar\omega_0 \ll KT$ the spacing betn the energy levels are much smaller than the total vol

$$\langle N \rangle = \frac{L}{2\pi\hbar} \int_0^\infty \frac{e^{-\beta[\hbar\omega_0(n+\frac{1}{2})-\mu]}}{1-e^{-\beta[\hbar\omega_0(n+\frac{1}{2})-\mu]}}\, dn$$

$$1-e^{-\beta[\hbar\omega_0 n+\frac{\hbar\omega_0}{2}-\mu]} = x$$

$$= \frac{L}{2\pi\hbar}\frac{1}{\beta\hbar\omega_0}\left[\ln x\right]_{(1-e^{-\beta(\frac{\hbar\omega_0}{2}-\mu)})}^{1}$$

$$\Rightarrow \beta\hbar\omega_0\, e^{-\beta[\hbar\omega_0(n+\frac{1}{2})-\mu]} dn = d$$

$$n=0,\ x = 1-e^{-\beta(\frac{\hbar\omega_0}{2}-\mu)}$$

$$n=\infty,\ x = 1$$

$$\langle N \rangle = \frac{L}{(2\pi\hbar)} \frac{1}{\beta\hbar\omega_0} \ln\left[\frac{1}{1 - e^{-\beta(\hbar\omega_0/2 - \mu)}}\right]$$

$$= -\frac{L}{2\pi\hbar} \frac{1}{\hbar\omega_0} \frac{\ln\left(1 - \overbrace{e^{-\beta(\hbar\omega_0/2 - \mu)}}^{\ll 1}\right)}{\beta}$$

b)

* In order to check for phase transition (condenstate) we set μ at the ground state & $T=0$ to see if there is any restriction on $\langle N \rangle$. If there is no restriction then there is no phase transition

Solving for μ

$\ln(1+x) = x - \frac{x^2}{2} \cdots$

$$N = -\frac{L}{2\pi\hbar} \frac{1}{\beta\hbar\omega_0} \left(-e^{-\beta(\hbar\omega_0/2 - \mu)}\right)$$

$$\frac{2\pi\hbar}{L} N\beta\hbar\omega_0 = e^{-\beta\left(\frac{\hbar\omega_0}{2} - \mu\right)}$$

$$\ln\left(\frac{2\pi\hbar}{L}\right) + \ln N + \ln\beta + \ln\hbar\omega_0 = -\beta\hbar\omega_0 + \beta\mu$$

$$\mu = \frac{\ln(2\pi\hbar^2\omega_0)}{\beta} + \frac{\ln N}{\beta} + \frac{\ln\beta}{\beta} + \hbar\omega_0$$

At, $T \to 0$ $\beta \to \infty$ & $\mu = \hbar\omega_0$

then there is no restriction on N

Hence there is no phase transition

$$\langle N \rangle = \left(\frac{L}{2\pi}\right)^2 \int_0^\infty (n+1) \frac{e^{-\beta[\hbar\omega_0(n+1)-\mu]}}{1 - e^{-\beta[\hbar\omega_0(n+1)-\mu]}} \, dn$$

$$= \int_0^\infty \frac{(n+1) \quad dn}{e^{\beta[\hbar\omega_0(n+1)-\mu]} - 1} \qquad + \frac{1}{e^{\beta(\hbar\omega_0 - \mu)}} = \frac{1}{z^{-1} e^{\beta\hbar\omega_0}}$$

$$z = e^{\beta\mu}$$

$$= \int_0^\infty \frac{(n+1) dn}{z^{-1} e^{\beta[\hbar\omega_0(n+1)]} - 1}$$

$$\beta\hbar\omega_0 (n+1) = x \quad \Rightarrow (n+1) = \frac{x}{\beta\hbar\omega_0}$$

$$\Rightarrow \beta\hbar\omega_0 \, dn = dx$$

$$n = 0, \; x = \beta\hbar\omega_0$$

$$n = \infty, \; x = \infty$$

$$= \frac{1}{(\beta\hbar\omega_0)^2} \int_{\beta\hbar\omega_0}^\infty \frac{x\,dx}{z^{-1} e^x - 1}$$

$$= \frac{1}{(\beta\hbar\omega_0)^2} \int_0^\infty \frac{x\,dx}{z^{-1} e^x - 1} - \frac{1}{(\beta\hbar\omega_0)^2} \frac{z}{1-z}$$

For $\hbar\omega_0 \ll KT$ which is the classical limit

$z (= e^{\beta\mu}) \ll 1$ the 2nd term is negligible

So,

$$\langle N \rangle = \left(\frac{L}{2\pi}\right)^2 \frac{1}{(\beta\hbar\omega_0)^2} \underbrace{\int_0^\infty \frac{x^{2-1}\,dx}{z^{-1}e^x - 1}}_{= \underbrace{\Gamma(2)}_{=1} g_2(z)}$$

$\Gamma(2) = 1! = 1$

$$\langle N \rangle = \frac{L^2 \, g_2(z)}{(2\pi\hbar\beta\omega_0)^2}$$

d) At high temp we found

$$N_e = \frac{L^2}{(2\pi)^2} \frac{1}{(\hbar\beta\omega_0)^2} g_2(z)$$

$\uparrow$

# of particles in the excited state

as $T \to 0$, $z \to 1$

$$g_2(1) = \zeta(2)$$

Hence, as $T \to 0$ $g_2(z) \leq \zeta(2)$

which puts a restriction on the no. of particles, so if $N > \frac{L^2}{(2\pi)^2} \frac{1}{(\hbar\beta\omega_0)^2} \zeta(2)$

there is BEC and

$$T_c^2 = \frac{N\hbar^2\omega_0^2 (2\pi)^2}{K^2 L^2} \zeta(2) \qquad T_c = \frac{\sqrt{N}\, 2\pi\hbar\omega_0}{KL} \sqrt{\zeta(2)}$$

b) For a two dimensional Harmonian

$$E = \hbar\omega_0 (n+1) \Rightarrow \text{energy of 1 oscillator}$$

and $n = n_x + n_y$

the degeracy of the oscillator is,

$$g = \begin{pmatrix} \overset{\text{\# of dim}}{N} + n - 1 \\ n \end{pmatrix} = \begin{pmatrix} 2+n-1 \\ n \end{pmatrix} = \begin{pmatrix} n+1 \\ n \end{pmatrix}$$

$$= \frac{(n+1)!}{n!\,(n+1-n)} = \frac{(n+1)\,n!}{n!} = (n+1)$$

Then,

$$\langle N \rangle = \int_0^\infty (n+1) \left[\frac{e^{-\beta[\hbar\omega_0 (n+1) - \mu]}}{1 - e^{-\beta[\hbar\omega_0 (n+1) - \mu]}} \right] dn$$

$$= \underbrace{\int_0^\infty n [\;\;] \, dn}_{I} + \int_0^\infty [\;\;] \, dn$$

$$I = \int_0^\infty \frac{n\, e^{-\beta[\hbar\omega_0(n+1)-\mu]}}{1 - e^{-\beta[\hbar\omega_0(n+1)-\mu]}}\, dn$$

$$1 - e^{-\beta[\hbar\omega_0(n+1)-\mu]} = x$$

$$\Rightarrow \quad n\beta\hbar\omega_0\, e^{-\beta[\hbar\omega_0(n+1)-\mu]}\, dn = dx$$

$$\Rightarrow \quad n\, e^{-\beta[\hbar\omega_0(n+1)-\mu]}$$

$$\int_0^\infty \frac{n}{z^{-1} e^{\beta[\;]} - 1}\, dn$$

$$\beta\hbar\omega_0(n+1) = x$$

$$\Rightarrow \beta\hbar\omega_0\, dn = dx$$

$$\int_0^\infty \frac{x^n}{z^{-1}e^x - 1}$$