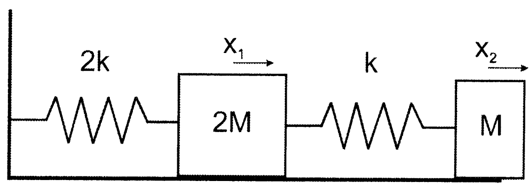
Mechanics and Statistical Mechanics Qualifying Exam
Fall 2006

Problem 1: (10 Points)

Two blocks are free to move in *one* dimension along a frictionless horizontal surface. The blocks of mass 2M and M are connected to each other and to a fixed wall by two springs with stiffness 2k and k as shown in the figure. Choose the dynamical coordinates of the system to be the position of block 1, x_1, and block 2, x_2, from their respective equilibrium positions. Consider only small oscillations so that the springs are linear. Neglect all damping.



a. Write down the equations of motion for each mass. **(2 Points)**

b. What are the normal mode oscillations that you expect to see? Just give the number of normal modes and the relative directions of the oscillations of the two masses for each normal mode. **(2 Points)**

c. Find the normal mode frequencies and eigenvectors that describe the normal modes. **(4 Points)**

d. Suppose you grab mass M and pull it slowly to the right by an amount A_0. When mass 2M is stationary show that it is $A_0/4$ from its equilibrium position. **(1 Points)**

e. If you release the system from the starting position in d., what will be the displacement of the system as a function of time? Write your answer in terms of the normal modes. **(1 Points)**

Problem 2 (10 Points):

Imagine a modification to Newton's universal law of gravitation such that there is an additional attractive term that varies inversely as the cube of the separation between two masses.

$$\vec{F} = (-\frac{k}{r^2} - \frac{\lambda}{r^3})\hat{\mathbf{r}},$$

where k and λ are constants. Take the reduced mass as μ.

a. Find the potential energy assuming $U(r) \to 0$ as $r \to \infty$. **(1 Points)**

b. Show that the angular momentum $\vec{\mathbf{L}}$ of one mass about the other is conserved. **(1 Points)**

c. Sketch a graph of the effective potential energy as a function of r. **(1 Points)**

d. For a given value of the angular momentum, what is the radius of a circular orbit? Assume that $L^2/2\mu > \lambda$. **(2 Points)**

e. Under what conditions is the circular orbit in d. stable? **(2 Points)**

f. What is the frequency of radial oscillations about the circular orbit you found in d. if the system is slightly perturbed? **(2 Points)**

g. If λ can take any value, what condition in terms of the constants that are given and the reduced mass assures that there will be no stable circular orbits? **(1 Points)**

2.

a) $$\bar{F} = \left(-\frac{K}{r^2} - \frac{\lambda}{r^3}\right)\hat{r}$$

$$\Rightarrow \quad \bar{F} = -\frac{\partial U}{\partial r}\hat{r}$$

$$\Rightarrow \quad U = -\int F\,dr$$

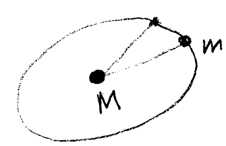
$$\Rightarrow U(r) = -\frac{K}{r} - \frac{\lambda}{2r^2} + \text{const}$$

For $r = \infty$, $V(r) = 0$

& const $= 0$

$$\Rightarrow \quad U(r) = -\frac{K}{r} - \frac{\lambda}{2r^2}$$

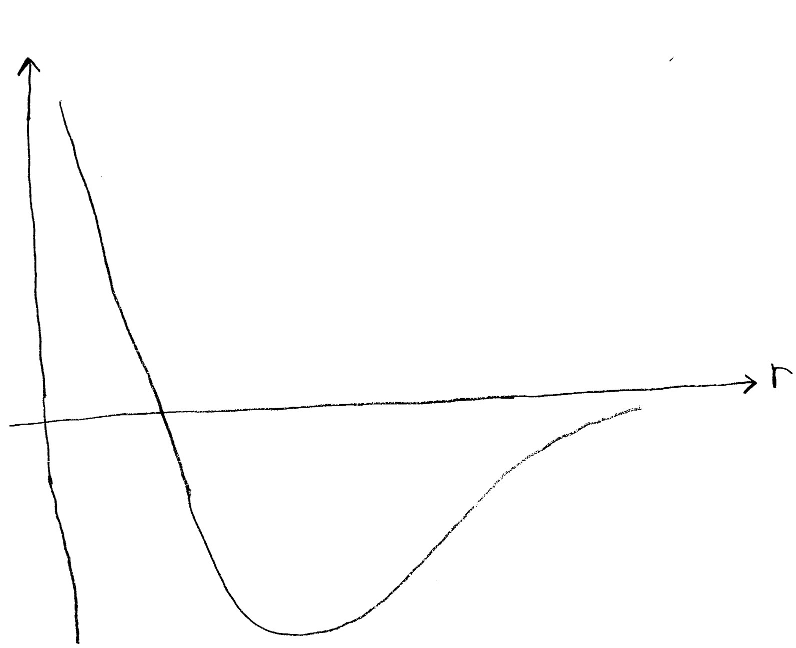
b) $L = \frac{1}{2}\mu r^2\dot{\theta}^2 + \frac{K}{r} + \frac{\lambda}{r} \Rightarrow$ lag for the 2nd mass,



θ is cyclis,

$\dot{p}_\theta = 0 \quad \Rightarrow$ angular momentum is const.

c)



$$p_\theta = \mu r^2 \dot{\theta}$$

d) $$H = \frac{p_r^2}{2\mu r^2} + \underbrace{\frac{1}{2}\mu^2 r^2 \dot{\theta}}_{= \frac{p_\theta^2}{2\mu r^2}} - \frac{K}{r} - \frac{\lambda}{2r^2}$$

$$\dot{\theta} = \frac{\partial H}{\partial \theta} = \frac{p_\theta}{\mu r^2}$$

$$\dot{r} = \frac{\partial H}{\partial p_r} = -\frac{p_r}{\mu r^2}$$

$$\dot{p}_r = -\frac{\partial H}{\partial r} = -\frac{K}{r^2} - \frac{\lambda}{r^3} + \frac{p_\theta^2}{\mu r^3}$$

$$\ddot{r} = -\frac{K}{\mu r^4} - \frac{\lambda}{\mu r^5} + \frac{P_\theta^2}{\mu r^5}$$

For circular orbit $\ddot{r} = 0$

$$0 = -\frac{Kr}{\mu} - \frac{\lambda}{\mu} + \frac{P_\theta^2}{\mu^2}$$

$$\Rightarrow Kr = \frac{P_\theta^2}{\mu} - \lambda$$

$$\Rightarrow r = \frac{1}{K}\left(\frac{P_\theta^2}{\mu} - \lambda\right)$$

e) For equilibrium

$$\frac{\partial V_{eff}}{\partial r} = 0$$

Now, if $\frac{\partial^2 V_{eff}}{\partial r^2} > 0 \Rightarrow$ stable equilibrium

$$V_{eff} = \frac{P_\theta^2}{2\mu r^2} - \frac{K}{r} - \frac{\lambda}{2r^2}$$

$$\frac{\partial V_{eff}}{\partial r} = 0$$

$$\Rightarrow -\frac{P_\theta^2}{\mu r^3} + \frac{K}{r^2} + \frac{\lambda}{r^3} = 0$$

$$\Rightarrow -\frac{P_\theta^2}{\mu} + Kr + \lambda = 0$$

$\Rightarrow r = \frac{1}{K}\left(\frac{P_\theta^2}{\mu} - \lambda\right)$

another way of getting radius!

f) $$L = \frac{1}{2}\mu \dot{r}^2 + \frac{1}{2}\mu r^2 \dot{\theta}^2 + \frac{K}{r} + \frac{\lambda}{r}$$

$$\tilde{T} = \mu \qquad R = \frac{1}{2}\mu \dot{r}^2 - \frac{P_\theta^2}{2\mu r^2} + \frac{K}{r} + \frac{\lambda}{r}$$

$$V_{eff} = \frac{P_\theta^2}{2\mu r^2} - \frac{K}{r} - \frac{\lambda}{r}$$

$$\frac{1}{2}\frac{\partial^2 V_{eff}}{\partial r^2}\bigg|_{r=r_{eq}} r^2 = \frac{1}{2}\frac{\partial}{\partial r}\left\{-\frac{P_\theta^2}{\mu r^3} + \frac{K}{r^2} + \frac{\lambda}{r^3}\right\} r^2 \bigg|_{r=r_{eq}}$$

$$= \frac{1}{2}\left\{\frac{3P_\theta^2}{\mu r^4} - \frac{2K}{r^3} - \frac{3\lambda}{r^4}\right\} r^2 \bigg|_{r=r_{eq}} \qquad r_{eq} = \frac{1}{K}\underbrace{\left(\frac{P_\theta^2}{\mu} - \lambda\right)}_{= \alpha}$$

$$= \frac{1}{2}\left\{3\left(\frac{P_\theta^2}{\mu} - \lambda\right)\frac{1}{r^4} - \frac{2K}{r^3}\right\} r^2 \qquad = \frac{\alpha}{K}$$

$$= \frac{1}{2}\left\{3\frac{K^4}{\alpha^3} - 2\frac{K^4}{\alpha^3}\right\} r^2 = \frac{1}{2}\frac{K^4}{\alpha^3} r^2$$

$$\det |\tilde{V} - \lambda \tilde{T}| = 0$$

$$\Rightarrow \lambda = \frac{K^4}{\mu \alpha^3} \Rightarrow \omega = \frac{K^2}{\mu^{1/2}\alpha^{3/2}}$$

9) For unstable orbit

$$\frac{\partial^2 V_{eff}}{\partial r^2} < 0$$

$$\Rightarrow \quad 3r\left(\frac{P_\theta^2}{\mu} - \lambda\right) < 2K$$

$$\Rightarrow \quad \frac{P_\theta^2}{\mu} - \lambda < \frac{2K}{3r}$$

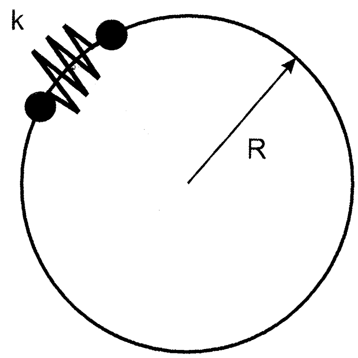
$$\Rightarrow \quad -\lambda < \frac{2K\mu}{3r P_\theta^2}$$

$$\Rightarrow \quad \lambda > \frac{2K\mu}{3r P_\theta^2}$$

or $r > \frac{3}{2K}\left(\frac{P_\theta^2}{\mu} - \lambda\right)$

Problem 3 (10 Points):

Two beads of mass m are connected to each other with a massless spring with spring constant k. The beads are free to move on a circular wire of radius R. The unstretched length of the spring is $R\phi_0$. Use the generalized coordinates $\theta =$ angular orientation of the midpoint between the two masses (the center of the spring) and $\phi =$ angular distance between the two masses to answer the following questions. Neglect gravity. The restoring force of the spring is proportional to ϕ.



a. Find the kinetic energy of the system. **(1 Points)**

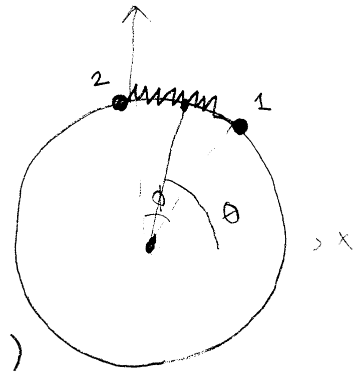
b. Write down the potential energy of the system. **(1 Points)**

c. Find the Lagrangian of the system. **(2 Points)**

d. Solve for $\theta(t)$ and $\phi(t)$. **(4 Points)**

e. Is $H = T + U$? Why? You must explain your answer in words to receive any credit. **(2 Points)**

3.



a) $$T = \frac{1}{2}m(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m(\dot{x}_2^2 + \dot{y}_2^2)$$

$$x_1 = R\cos(\theta - \tfrac{\phi}{2}) \Rightarrow \dot{x}_1 = -(\dot{\theta} - \tfrac{\dot{\phi}}{2})R\sin(\ )$$

$$Y_1 = R\sin(\theta - \tfrac{\phi}{2}) = \dot{Y}_1 = (\dot{\theta} - \tfrac{\dot{\phi}}{2})R\cos(\ )$$

$$x_2 = R\cos(\theta + \tfrac{\phi}{2}) \Rightarrow \dot{x}_2 = -(\dot{\theta} + \tfrac{\dot{\phi}}{2})R\sin(\theta + \tfrac{\phi}{2})$$

$$Y_2 = R\sin(\theta + \tfrac{\phi}{2}) \Rightarrow \dot{Y}_2 = (\dot{\theta} + \tfrac{\dot{\phi}}{2})R\cos(\theta + \tfrac{\phi}{2})$$

$$T = \frac{1}{2}mR^2(\dot{\theta} - \tfrac{\dot{\phi}}{2})^2 + \frac{1}{2}mR^2(\dot{\theta} + \tfrac{\dot{\phi}}{2})^2$$

$$= \frac{1}{2}mR^2\left(2\dot{\theta}^2 + 2\frac{\dot{\phi}^2}{4}\right)$$

$$= mR^2\left(\dot{\theta}^2 + \frac{\dot{\phi}^2}{4}\right)$$

b) $$V = \frac{1}{2}Ks^2 = \frac{1}{2}KR^2(\phi - \phi_0)^2$$

c) $$L = T - V = mR^2\left(\dot{\theta}^2 + \frac{\dot{\phi}^2}{4}\right) - \frac{1}{2}KR^2(\phi - \phi_0)^2$$

d)

$$P_\theta = \frac{\partial L}{\partial \dot\theta} = 2mR^2\dot\theta$$

$$P_\phi = \frac{1}{2} mR^2 \dot\phi$$

$$H = mR^2\dot\theta^2 + \frac{1}{4} mR^2\dot\phi^2 + \frac{1}{2}KR^2(\phi-\phi_0)^2$$

$$= \frac{P_\theta^2}{4mR^2} + \frac{P_\phi^2}{mR^2} + \frac{1}{2}KR^2(\phi-\phi_0)^2$$

$$\dot\phi = \frac{\partial H}{\partial P_\phi} = \frac{2P_\phi}{mR^2}$$

$$\dot P_\phi = -\frac{\partial H}{\partial \phi} = KR^2(\phi-\phi_0)$$

$$\ddot\phi = \frac{2}{mR^2}\dot P_\phi = -\frac{2K}{m}(\phi-\phi_0)$$

$$\Rightarrow \quad \phi_c(t) = A\left(e^{i\sqrt{2K/m}\,t} + e^{-i\sqrt{2K/m}\,t}\right)$$

$$\phi_p = \phi_0$$

say, initialy (i.e $t=0$) $\phi = \phi_0$

$$\phi_0 = 2A \quad \Rightarrow A = \phi_0/2$$

$$\Rightarrow \phi(t) = \frac{\phi_0}{2}\left[e^{+i\sqrt{2K/m}\,t} + e^{-i\sqrt{2K/m}\,t}\right] + \phi_0$$

$$\ddot{\phi} + \frac{2K}{m}\phi = \frac{2K}{m}\phi_0$$

$$\rightsquigarrow \quad \ddot{\phi} + \frac{2K}{m}\phi = 0$$

$$\lambda = \pm i\sqrt{\frac{2K}{m}} = \pm i\omega$$

$$\phi_c(t) = A\cos\omega t + B\sin\omega t$$

$$\rightsquigarrow \quad \text{let, } \phi_p(t) = Ct^2 + Dt + E$$

$$\ddot{\phi}_p = 2ct$$

Thus,

$$2ct + \frac{2K}{m}(ct^2 + Dt + E) = \frac{2K}{m}\phi_0$$

$$\Rightarrow c = 0$$

$$\Rightarrow D = 0$$

$$\Rightarrow E = \phi_0$$

$$\therefore \phi_p(t) = \phi_0$$

$$\Rightarrow \phi(t) = A\cos\omega t + B\sin\omega t + \phi_0$$

At, $t=0$, $\phi(0)=\phi_0$

$$\Rightarrow A = 0$$

$$\Rightarrow \phi(t) = B\sin\omega t + \phi_0 \quad ; \quad \omega = \sqrt{\frac{2K}{m}}$$

$$\ddot{\theta} = 0$$

$$\theta(t) = At + B$$

$$\theta(t) = \theta_0 + At$$

e) Since $\phi \equiv \phi(t)$ & $\theta \equiv \theta(t)$, the L depends out

So, $H \neq T + U = E$

as, $h = \underbrace{\frac{1}{4} mR^2 \dot{\phi}^2}_{T_2} + \underbrace{\frac{p_\theta^2}{4mR^2}}_{T_0} + \underbrace{\frac{1}{2} KR^2 (\phi - \phi_0)^2}_{V}$

$$h = E = T_2 + V$$

Problem 4 (10 Points):

This problem involves the mean field Ising model. Consider a solid containing N electrons localized at lattice sites. Each electron has a magnetic moment μ. In a magnetic field H each electron can exist in one of two states, with energies $\pm\mu\,\mathrm{H}$.

a. Show that for *non-interacting electrons* the total magnetic moment is given by $\mathrm{M} = \mathrm{N}\,\mu\tanh(\frac{\mu H}{kT})$. **(2 Points)**

b. In order to add interactions between the electrons, assume that each electron sees an effective magnetic field equal to the applied field plus a local field arising from its neighbors. In this case, $\mathrm{H}_{eff} = \mathrm{H} + \frac{\alpha}{N}\,\mathrm{M}$, where α is a positive constant. Write down a self consistency equation that determines M. **(2 Points)**

trancendental eqn

c. Show that there is a spontaneous magnetization (e.g. when $\mathrm{H} = 0$) below some critical temperature, T_c, and determine its value. **(3 Points)**

d. Show that the magnetic susceptibility χ, diverges at $\mathrm{T} \rightarrow \mathrm{T}_c$ from the high T side. (Hint: be careful because you will have to take a derivative of a transcendental equation.) **(3 Points)**

4.

a)

$$\epsilon = \pm \mu H$$

$$Z_1 = \sum_i e^{-\beta\epsilon_i} = e^{-\beta\mu H} + e^{+\beta\mu H}$$

$$Z = Z_1^N = \left(e^{\beta\mu H} + e^{-\beta\mu H}\right)^N =$$

$$F = -KT \ln Z = -2NKT\ln(\cosh(\beta\mu H))$$

$$M = -\frac{\partial F}{\partial H} = -2NKT\frac{(-\sinh(\beta\mu H))}{\cosh(\beta\mu H)}\mu\beta$$

$$M = N\mu_0 \tanh\left(\frac{\mu H}{KT}\right)$$

b)

$$\epsilon = \pm \mu H_{eff} = \pm\mu\left(H + \frac{\alpha}{N}M\right)$$

$$M = N\mu_0 \tanh\left(\mu_0\beta H + \mu_0\beta\frac{\alpha}{N}M\right)$$

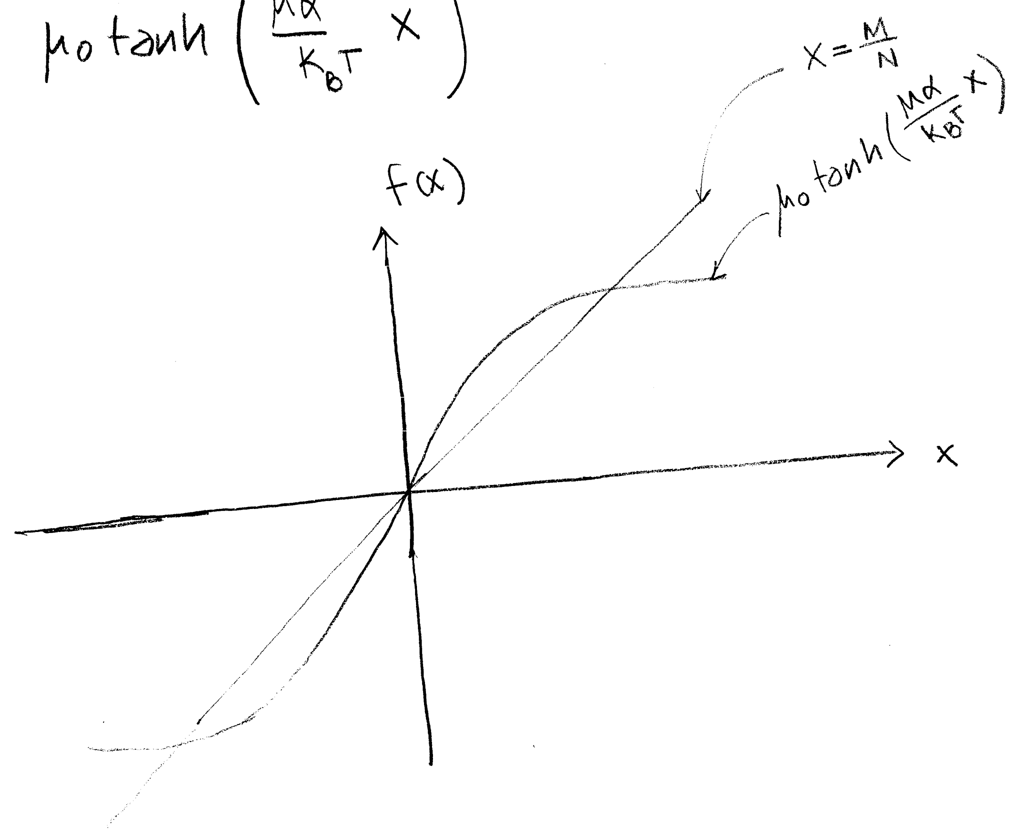
c) At, $H = 0$

$$M = N\mu_0 \tanh\left(\frac{\mu\alpha}{N K_B T} M\right)$$

$$\Rightarrow \quad \frac{M}{N} = \mu_0 \tanh\left(\frac{\mu\alpha}{K_B T} \frac{M}{N}\right)$$

$$X \equiv \frac{M}{N}$$

$$\Rightarrow \quad \underset{\substack{\uparrow \\ \text{linear} \\ \text{slope}=1}}{X} = \mu_0 \tanh\left(\frac{\mu\alpha}{K_B T} X\right)$$



$$f(x) = \mu_0 \tanh\left(\frac{\mu_0\alpha}{K_B T} x\right)$$

$$\left.\frac{df(x)}{dx}\right|_{x=0} = \frac{\mu_0^2\alpha}{K_B T} \underbrace{\operatorname{sech}^2(x)}_{=1}\Big|_{x=0} = \frac{\mu_0^2\alpha}{K_B T}$$

↑ as we need the slope of the linear section

* to have solution,

$$\frac{\mu_0^2 \alpha}{k_B T} \geq 1$$

At, $T = T_c$

$$\frac{\mu_0^2 \alpha}{k_B T_c} = 1$$

$$\Rightarrow \quad T_c = \frac{\mu_0^2 \alpha}{k_B}$$

As, $T < T_c$ the slope is steeper, and we have spontenous magnetization

d) * Now $M \propto H_0$ ← applied field

$$\Rightarrow M = \chi H_0$$

unlike when we calculated magnetization from free energy, as the free energy varies with total field $H = H_0 + M$

$$\chi = \frac{\partial M}{\partial H}$$

$$M = N\mu_0 \tanh\left(\mu_0\beta H + \mu_0 \frac{\beta\alpha}{N} M\right)$$

$$\frac{\partial M}{\partial H} = N\mu_0 \operatorname{sech}^2\left(\mu_0\beta H + \mu_0 \frac{\beta\alpha}{N} M\right)\left(\mu_0\beta + \frac{\mu_0\beta\alpha}{N}\frac{\partial M}{\partial H}\right)$$

/* $\Rightarrow \chi\left(1 - \mu_0^2\beta\alpha \operatorname{sech}^2()\right) = N\mu_0^2\beta \operatorname{sech}^2()$

$$T_c = \frac{\mu_0^2\alpha}{K_B}$$

$$\Rightarrow \chi = \frac{N\mu_0^2\beta \operatorname{sech}^2()}{1 - \mu_0^2\beta\alpha \operatorname{sech}^2()}$$

$$\chi\Big|_T = \frac{\frac{N\mu_0^2}{K_B}\frac{1}{T}\operatorname{sech}^2\left(\frac{\mu_0}{K_B}\frac{1}{T}H + \frac{\mu_0\alpha}{K_B N T}M\right)}{1 - \frac{\mu_0^2\alpha}{K_B}\frac{1}{T}\operatorname{sech}^2(\quad)}$$ */

$$\chi\left\{1 - \frac{\frac{\mu_0\beta\alpha}{N}}{\cosh^2()}\right\} = \frac{N\mu_0^2\beta}{\cosh^2()}$$

$$\Rightarrow \chi = \frac{N\mu_0^2\beta}{\cosh^2() - \frac{\mu_0\beta\alpha}{N}}$$

$$= \frac{\frac{N\mu_0^2}{K_B}\frac{1}{T}}{\cosh^2\left(\frac{\mu_0}{K_B}\frac{1}{T}H + \frac{\mu_0\alpha}{K_B N}\frac{M}{T}\right) - \frac{\mu_0\beta\alpha}{N}}$$

* doesn't go to ∞ At T_c !

At $T > T_c$ $M = 0$

$$\chi = \frac{N\mu_0^2 \beta}{\cosh^2(\mu\beta H) - \mu\beta\alpha}$$

set $H = 0$

Problem 5 (10 Points):

The distribution function for an ideal Bose gas is given by,

$$f(\vec{x}, \vec{p}) = g \; [\; e^{(\epsilon - \mu)/kT} \text{ - } 1 \;]^{-1}$$

a. Define all the quantities found in $f(\vec{x}, \vec{p})$. **(1 Points)**

b. What is the value of g for photons? **(1 Points)**

c. What is the meaning of the distribution function? Sketch the distribution as a function of energy. Make sure to label your sketch with the parameters. **(1 Points)**

d. For photons to be in thermal equilibrium there must be at least a small amount of matter present, since the interaction between photons is negligible. What processes bring the photons into equilibrium with the matter? **(1 Points)**

e. Use the information in part d. and the definition of chemical potential, $\mu = \partial F / \partial N |_{T,V}$ to explain why the chemical potential of photons must be zero. **(2 Points)**

f. Find the mean energy density of a photon gas in thermal equilibrium at temperature T. **(4 Points)**

Problem 6 (10 Points):

The partition function for an ideal gas of molecules in a volume V can be written as, $Z = \frac{1}{N!}(V\,\zeta)^N$,

where $V\,\zeta$ is the partition function for a single molecule (involving its kinetic energy plus internal energy if it is not monotonic) and ζ is a function that depends only on the absolute temperature.

When these molecules are condensed to form a liquid, the crudest approximation consists of treating the liquid as if the molecules still formed a gas of molecules moving independently provided that,

1. each molecule is assumed to have a constant potential energy $-\eta$ due to its average interaction with the rest of the molecules.
2. each molecule is assumed to move throughout a volume $N\,\nu_0$, where ν_0 is the constant volume available per molecule in the liquid phase.

a. With these assumptions and the equation given in the above text, write down the partition function for a liquid consisting of N_L molecules. **(2 Points)**

b. Using the equation in the above text, write down the chemical potential μ_G for N_G molecules of the vapor in a volume V_G at temperature T. Treat the system as an ideal gas. **(1 Points)**

c. Write down the chemical potential μ_l for N_l of the molecules of the liquid at the temperature T using the result from (a.). **(1 Points)**

d. Using your results in b and c, find an expression relating the vapor pressure to the temperature T where the gas is in equilibrium with the liquid. **(2 Points)**

e. Use the Clausius-Clayperon equation and the fact that the gas can be considered ideal to show that $P = P_0\, e^{-L/RT}$, where L in the latent heat of vaporization per mole. **(2 Points)**

f. Calculate the molar entropy difference between the gas and liquid in equilibrium at the same temperature and pressure. **(2 Points)**

6.

The total partition func for a gas in a vol. V is

$$Z_G = \frac{1}{N!}(V\xi)^N$$

where, $Z_{1G} = V\xi$

a) For a liquid,

the one particle Hamiltonian is

$$H = \underbrace{\frac{p^2}{2m}}_{H_0} - \underbrace{\eta}_{H_I}$$

the form of the one particle free-partition func would be same as the gas with a volume $N v_0$ thus,

$$Z_{1L}^{0} = (N v_0)\xi$$

and the partition func for the interaction part would be,

$$Z_{1L}^{I} = e^{\beta\eta}$$

So, the total one particle partition func is

$$Z_{1L} = Z^{0}_{1L}\, Z^{I}_{1L} = (N v_0)\xi\, e^{\beta\eta}$$

So, the total partition func is

$$Z_L = \frac{1}{N!}\left[(N v_0)\xi\, e^{\beta\eta}\right]^N$$

b)

$$\mu_G = \left.\frac{\partial F}{\partial N_G}\right|_{V,T}$$

$$= -KT \frac{\partial \ln Z_G}{\partial N_G}$$

$$= -KT \frac{\partial}{\partial N_G}\left\{ N_G \ln (V_G \xi) - N_G \ln N_G + N_G \right\}$$

$$= -KT \left\{ \ln (V_G \xi) - \ln N_G \right\}$$

c) $\mu_L = \left.\frac{\partial F}{\partial N_G}\right|_{V,T}$

$$= -KT \frac{\partial}{\partial N_L} \ln Z_L$$

$$= -KT \frac{\partial}{\partial N_L} \left\{ N_L \ln\left(N_L v_0 \xi e^{\beta\eta}\right) - N_L \ln N_L + N_L \right\}$$

$$= -KT \left\{ \ln\left(v_0 \xi e^{\beta\eta}\right) + 1 \right\}$$

d) At equilibrium

$$\mu_G = \mu_L$$

$$\ln\left(\frac{V_G \xi}{N_G}\right) = \ln\left(v_0 \xi e^{\beta\eta} e^{1}\right)$$

$$\Rightarrow \frac{V_G \xi}{N_G} = v_0 \xi e^{\beta\eta + 1}$$

$$\Rightarrow \frac{V_G}{N_G} = v_0 e^{\beta\eta + 1}$$

Now,

$$P = - \frac{\partial F}{\partial V}\Big|_{N,T}$$

$$P_G = + KT \frac{\partial}{\partial V_G} \ln Z_G$$

$$= KT \frac{\partial}{\partial V_G} \left\{ N_G \ln (V_G \xi) \right\}$$

$$= KT \frac{N_G}{V_G \xi} \xi$$

$$P_G = KT \left(\frac{N_G}{V_G} \right)$$

$$\Rightarrow P_G = \frac{KT}{v_0 e^{\beta\eta+1}}$$

e) The Clausius – Clapeyron Relation is

latent heat for converting the material from liquid to gas

$$\frac{dP}{dT} = \frac{L}{T\,\Delta V}$$ where, $\Delta V = V_G - V_L$

$P \rightarrow$ pressure at equilibrum betn the liq & gaseous phase

Now, $V_L \ll V_G$ so, $V_G - V_L \approx V_G$

can be taken const over a small section of the P-T curve

$$\frac{dP}{dT} = \frac{L}{T V_G}$$

considering Ideal gas

$$PV_G = nRT$$

$$V_G = \frac{nRT}{P}$$

$$\frac{dP}{dT} = \frac{PL}{nRT^2}$$

$$\Rightarrow \frac{dP}{P} = \frac{L}{nR}\frac{dT}{T^2}$$

$$\Rightarrow \ln P = -\frac{L}{nRT}$$

$$\Rightarrow P = \text{const}\; e^{-L/nRT}$$

$$\Rightarrow P = P_0\, e^{-L/RT} \quad (n=1)$$

f)

* This can be seen by noting, At equi betn gas & liquid phase the free energies are equal at the phase boundary

Now,

$$\frac{dP}{dT} = \frac{S_G - S_L}{V_G - V_L} \;^{*}$$

At, equilibrium

$$P_G = \frac{KT}{v_0 e^{\eta/KT} + 1}$$

$$\frac{dP_G}{dT} = \frac{K(v_0 e^{\eta/KT}+1) - KT\left(\frac{v_0 \eta}{K}\left(-\frac{1}{T^2}\right)e^{\eta/KT}\right)}{(v_0 e^{\eta/KT}+1)^2}$$

$$= \frac{Kv_0(e^{\beta\eta}+1) + K\beta v_0 \eta e^{\beta\eta}}{(v_0 e^{\eta/KT}+1)^2}$$

$$= \frac{Kv_0(e^{\beta\eta} + \eta\beta e^{\beta\eta} + 1)}{(v_0 e^{\eta/KT}+1)^2}$$

Thus, $S_G - S_L = (V - N_L v_0) \dfrac{Kv_0(e^{\beta\eta} + \eta\beta e^{\beta\eta} + 1)}{(v_0 e^{\beta\eta}+1)^2}$