Mechanics and Statistical Mechanics Qualifying Exam
Fall 2005

Problem 1: (10 Points)

We can use the temperature rise that results from the adiabatic compression of an ideal monoatomic gas to measure the velocity of a bullet. Suppose a piston of mass M can move in a uniform frictionless tube of cross-sectional area A. The piston can only move in the direction of compression. The tube is closed at one end, and the piston is sealed so that no gas can escape. The cylinder is filled with He gas at temperature T_0 and pressure P_0, such that the initial position of the piston is L_0 from the closed end. A bullet of mass m is fired from a gun and strikes the center of the piston. The bullet embeds itself in the piston, causing the piston to move and compress the gas in the tube. The maximum temperature of the gas in the cylinder is T_f. Assume that the piston compresses the gas adiabatically.

a. Find the initial velocity of the bullet, v_0, in terms of the given parameters. **(2 Points)**

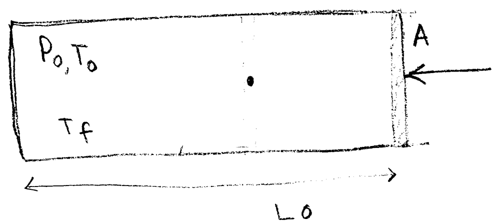
b. What is the maximum displacement of the piston, ΔL, in terms of the given parameters? **(2 Points)**

c. What is the maximum final pressure inside the cylinder, P_f, in terms of the given parameters? **(2 Points)**

d. Sketch the acceleration of the piston versus ΔL beginning at the moment the bullet hits the piston. Make sure that the sketch is *qualitatively* accurate. **(2 Points)**

e. Neglecting the exact time that the bullet impacts the piston, at what value of ΔL is the piston at when the magnitude of its acceleration is greatest?**(1 Points)**

f. We assumed that the gas was compressed adiabatically. If heat was lost to the walls of the cylinder, would the resulting value of v_0 be: (1.) too high, (2.) too low, or (3.) unchanged. To receive credit you must explain your answer. **(1 Points)**



a) * Adiabatic compression

$$\Delta E_{int} = Q + W \quad , \quad Q = 0$$

$$W = \Delta E_{int} = \frac{3}{2} NK (T_f - T_0)$$

* but work done is change in Kinetic energy of the piston

$$W = \Delta K$$

* the collision is inelastic

$$m v_0 = (M+m) v_1 \Rightarrow v_1 = \frac{m}{M+m} v_0$$

$$W = |\Delta K| = \frac{1}{2} (M+m) v_1^2 - 0$$

$$= \frac{1}{2} \frac{m^2 v_0^2}{M+m}$$

$$\rightsquigarrow \frac{3}{2} NK (T_f - T_0) = \frac{1}{2} \frac{m^2 v_0^2}{M+m}$$

$$\Rightarrow v_0 \quad \frac{\sqrt{3NK(T_f - T_0)}}{m}$$

b) $$W = \int P\,dV = \int P A\,dx$$

$$V T^{f/2} = \text{const}$$

$$\Rightarrow V_0 T_0^{3/2} = V_1 T_1^{3/2}$$

$$\Rightarrow V_1 = \frac{V_0 T_0^{3/2}}{T_1^{3/2}}$$

$$\Delta V = V_1 - V_0 = -V_0 \left(\frac{T_0^{3/2}}{T_f^{3/2}} - 1 \right) = -A \Delta L$$

As, $L_f = L_0 - \Delta L$

$$\Delta L = -L_0 \left(\frac{T_0^{3/2}}{T_f^{3/2}} - 1 \right) = L_0 \left(1 - \frac{T_0^{3/2}}{T_f^{3/2}} \right)$$

c) $$P_0 V_0^{\gamma} = P_1 V_1^{\gamma} \qquad \gamma = \frac{f+2}{f}$$

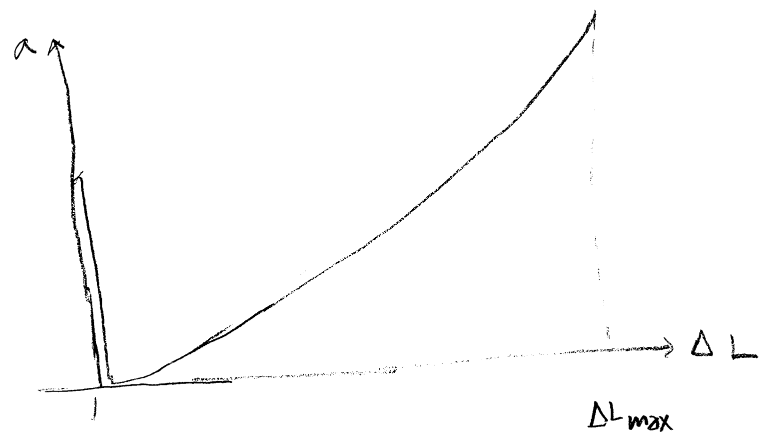
$$\Rightarrow P_1 = \frac{P_0 V_0^{5/3}}{V_0^{5/3} \left(\frac{T_0^{3/2}}{T_f^{3/2}} \right)^{5/3}}$$

$$\Rightarrow P_f = P_0 \left(\frac{T_0^{3/2}}{T_f^{3/2}} \right)^{-5/3} = P_0 \frac{T_f^{5/2}}{T_0^{5/2}}$$

d) $\Rightarrow P_f = P_o \left(1 - \frac{\Delta L}{L_o}\right)^{-5/3} = P_o \left(\frac{L_o}{L_o - \Delta L}\right)^{5/3}$

$\Rightarrow F = A P_o \left(\frac{L_o}{L_o - \Delta L}\right)^{5/3}$

$a \sim \left(\frac{1}{1 - \frac{\Delta L}{L_o}}\right)^{5/3}$



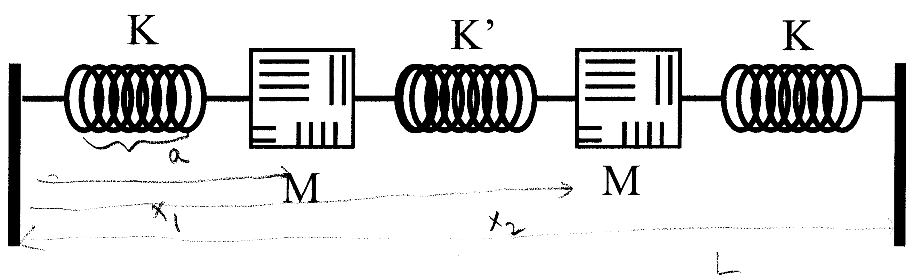
e) $\Delta L = \Delta L_{max}$

f) If heat was lost to the the walls of the cylinder the resulting value of V_o would be low as,

$W = \Delta E_{in} - Q$ would be smaller

Problem 2 (10 Points):

Identical objects of equal mass M are attached to posts by springs with spring constant K. A spring of spring constant K' is connected between the two masses. The springs have negligible mass. Treat the system in 1 dimension in space (horizontal dimension shown in the figure).



a. Derive differential equations that describe the motion of the two masses. **(2 Points)**

b. What are the normal modes for the system? Indicate the modes on a diagram with arrows on your answer sheet. **(2 Points)**

c. Find the frequencies of each of the normal modes. **(3 Points)**

d. Determine the functions that describe the motion of the masses if the initial velocities of both masses are zero and the displacement of mass 1 is L and mass 2 is -L. **(2 Points)**

e. How is the motion of the masses affected if the coupling is weak, K'<<K? How is the motion of the masses affected when K'>>K? **(1 Points)**

2. $L = \frac{1}{2}M\dot{x}_1^2 + \frac{1}{2}M\dot{x}_2^2 + \frac{1}{2}K(x_1-a)^2 + \frac{1}{2}K'(x_2-x_1-a)^2$

$+ \frac{1}{2}K\underbrace{(L-x_2-a)^2}_{= (2a-x_2)^2}$

$\rightsquigarrow L = 3a$

d) $K' \gg K$ symmetric motion, move together

$K'' \ll K$ antisymmetric motion

Problem 3 (10 Points):

Consider a sphere of mass M and radius R that rolls, without sliding, down a triangular wedge of mass m. The wedge is free to move on a horizontal frictionless surface. The moment of inertia of the sphere is $\frac{2}{5}MR^2$, and the incline of the wedge makes an angle ϕ with the horizontal surface.

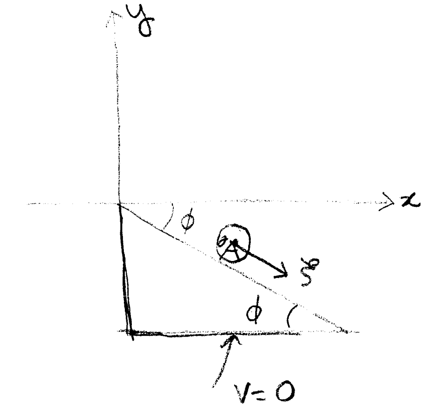
a. Find generalized coordinates for the system and label them on a sketch. **(2 Points)**

b. Find a Lagrangian that describes the system. **(4 Points)**

c. Calculate the acceleration of the wedge as the ball rolls down it. **(4 Points)**

3.

$$L = \frac{1}{2} M \dot{X}^2 + \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) + \frac{1}{2} I \dot{\theta}^2 - mgy$$



$$x = X + \xi \cos\phi \Rightarrow \dot{x} = -\dot{X} + \dot{\xi} \cos\phi$$

$$y = -\xi \sin\phi \Rightarrow \dot{y} = -\dot{\xi} \sin\phi$$

$$L = \frac{1}{2} M \dot{X}^2 + \frac{1}{2} m (-\dot{X} + \dot{\xi} \cos\phi)^2 + \frac{1}{2} m \dot{\xi}^2 \sin^2\phi + \frac{1}{2} I \dot{\theta}^2 + mg\xi \sin\phi$$

$$= \frac{1}{2} (M+m) \dot{X}^2 - m\cos\phi\, \dot{X} \dot{\xi} + \frac{1}{2} m \dot{\xi}^2 + \frac{1}{2} I \dot{\theta}^2 + mg\xi \sin\phi$$

condition for rolling without slipping

$$\xi = R\theta$$

$$\Rightarrow \dot{\xi} = R\dot{\theta} \Rightarrow \dot{\theta} = \frac{\dot{\xi}}{R}$$

$$= \frac{1}{2} (M+m) \dot{X}^2 - m\cos\phi\, \dot{X} \dot{\xi} + \frac{7}{10} m \dot{\xi}^2 + mg\xi \sin\phi$$

$$\rightsquigarrow (M+m) \ddot{X} - m\cos\phi\, \ddot{\xi} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\xi}}\right) - \frac{\partial L}{\partial \xi} = 0$$

$$\frac{7}{5} m \ddot{\xi} - m \ddot{X} \cos\phi - mg \sin\phi = 0$$

$$\Rightarrow \quad \ddot{\xi} = \frac{5}{7}\left(\ddot{X}\cos\phi + g\sin\phi\right)$$

Thus,

$$(M+m)\ddot{X} - \frac{5}{7} m \cos^2\phi \, \ddot{X} - \frac{5}{7} mg \sin\phi\cos\phi = 0$$

$$\ddot{X} = \frac{\frac{5}{7} mg \sin\phi \cos\phi}{M + m\left(1 - \frac{5}{7}\cos^2\phi\right)}$$

Problem 4 (10 Points):

A closed system consists of two distinguishable spin 1 magnets. Each magnet can have one of three orientations, $\uparrow$, $\leftrightarrow$, and $\downarrow$, with respect to the z axis. The respective magnetic moments are +m, 0 and -m. There is no applied field. The Hamiltonian, $H = B\ \Sigma m_i$.

a. List all the possible microstates of the system. What is the total number of states? **(1 Points)**

b. For B=0 what is the probability that the total magnetic moment, M, of the system is zero? **(1 Points)**

c. For B=0 compute average value of the total magnetic moment, $\langle M\rangle$, using the list in part (a.). **(1 Points)**

d. If $\Delta M = M - \langle M\rangle$, show that $(\Delta M)^2 = \langle M^2\rangle - \langle M\rangle^2$, and compute $(\Delta M)^2$ for B=0. **(2 Points)**

e. If the spins were *indistinguishable*, what would be the total number of microstates of the system? **(1 Points)**

For the last two parts of this problem consider N of the spins described in the initial part of the problem. These N spins are now in contact with a heat bath at temperature, T, and $B \neq 0$.

f. Find the partition function of the N spins. **(2 Points)**

g. What is the Helmholtz free energy of the N spins? **(2 Points)**

4.

a) 2 spin 1 magnets

1	2	
↑	↑	$= 2m$
↑	↔	$= m$
↑	↓	$= 0$
↔	↑	$= m$
↔	↔	$= 0$
↔	↓	$= -m$
↓	↑	$= 0$
↓	↔	$= -m$
↓	↓	$= -2m$

$$H = B \sum m_i$$

total no of states $= 5$

b) when $B = 0$

$$P(M=0) = \frac{3}{9} = \frac{1}{3}$$

c) $$\langle M \rangle = \frac{\sum M e^{-\beta E}}{\sum e^{-\beta E}} = \frac{0}{9} = 0$$

d) $$(\Delta M)^2 = \langle (M - \langle M \rangle)^2 \rangle$$

$$= \langle M^2 - 2M\langle M \rangle + \langle M \rangle^2 \rangle$$

$$= \langle M^2 \rangle - 2\langle M \rangle^2 + \langle M \rangle^2$$

$$= \langle M^2 \rangle - \langle M \rangle^2$$

$$\langle M^2 \rangle = \frac{12 m^2}{9} \qquad (\Delta M)^2 = \frac{12 m^2}{9}$$

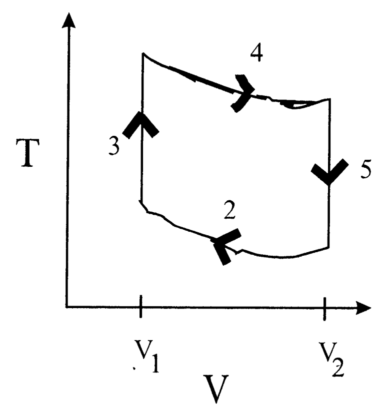
e) 6

f) $Z_1 = \sum_s e^{-\beta E_s}$

$=$

Problem 5 (10 Points):

The Otto cycle is shown in the figure. Stages 2 and 4 are adiabatic, reversible expansion and compression. Stages 3 and 5 are constant volume heating and cooling. Assume this is for an ideal gas.



a. Write down the efficiency, η, in terms of the work, W, and the added heat, Q. **(2 Points)**

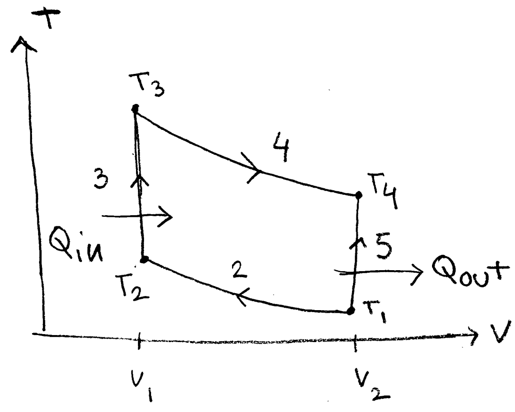
b. During which stage or stages is heat added? **(1 Points)**

c. Calculate the work, W, in terms of the heat capacity at constant volume, C_V and the temperature change. **(2 Points)**

d. Calculate the heat added, Q, in terms of C_V and the temperature change. **(2 Points)**

e. Show that the efficiency is η=1-$\frac{V_1}{V_2}^{\gamma-1}$, where γ=$\frac{C_p}{C_V}$. **(3 Points)**

5.



* Otto cycle is an internal combustion engine

Stage 2 & 4: adiabatic $Q=0$ $\Rightarrow T V^{\gamma-1} = \text{const}$

$\Delta U = W$

$T_1 V_2^{\gamma-1} = T_2 V_1^{\gamma-1}$

$T_3 V_1^{\gamma-1} = T_4 V_2^{\gamma-1}$

Stage 3 & 5: isochoric, $W=0$ $\Rightarrow PV = NKT$

$\Delta U = Q$

$\Rightarrow \frac{P_2 \cancel{V_1}}{T_2} = \frac{P_3 \cancel{V_1}}{T_3}$

$\frac{P_1 \cancel{V_2}}{T_1} = \frac{P_4 \cancel{V_2}}{T_4}$

$V_2 = V_2$

$\frac{NKT_1}{P_1} = \frac{NKT_4}{P_4}$

---

a) $\eta = \frac{W_{out}}{Q_{in}}$ $W_{out} = |Q_{in}| - |Q_{out}|$

b) Since, $Q=0$ for 2 & 4, the heat is added or removed in 3 & 5

$Q_3 = C_V \Delta T$; $\Delta T > 0$ $\therefore Q_3 > 0 \Rightarrow Q_{in}$

$Q_5 = C_V \Delta T$; $\Delta T < 0$ $\therefore Q_5 < 0 \Rightarrow Q_{OUT}$

c.

$$W = |Q_{in}| - |Q_{out}|$$

$$= |C_v(T_3 - T_2)| - |C_v(T_1 - T_4)|$$

$$= C_v \,|(T_3 - T_2) - (T_1 - T_4)|$$

d. $$Q_{in} = C_v(T_3 - T_2)$$

e. $$\eta = 1 - \frac{|Q_{out}|}{|Q_{in}|} = 1 - \frac{|T_1 - T_4|}{|T_3 - T_2|}$$

$$= 1 - \frac{(V_1/V_2)^{\gamma-1}\,|T_2 - T_3|}{|T_2 - T_3|}$$

$$= 1 - \left(\frac{V_1}{V_2}\right)^{\gamma-1}$$

$$\frac{T_1}{T_2} = \left(\frac{V_1}{V_2}\right)^{\gamma-1}$$

$$\frac{T_4}{T_3} = \left(\frac{V_1}{V_2}\right)^{\gamma-1}$$

$$T_1 = \left(\frac{V_1}{V_2}\right)^{\gamma-1} T_2$$

$$T_4 = \left(\frac{V_1}{V_2}\right)^{\gamma-1} T_3$$

$$\frac{T_2}{T_3} = \frac{P_2}{P_3}$$

$$\frac{T_4}{T_1} = \frac{P_4}{P_1}$$

$$\Rightarrow \frac{T_4}{T_3} \times \frac{T_2}{T_1} = \frac{P_4 P_2}{P_1 P_3}$$

$$\eta = 1 - \left(\frac{V_1}{V_2}\right)^{\gamma-1}$$

Problem 6 (10 Points):

Consider a white dwarf star that is composed of fully ionized ^{12}C and ^{16}O (a neutral plasma). The particle density of the star is uniform, and the electrons must be treated relativistically, E=pc.

a. Derive a relation between the Fermi energy of the electrons and the electron density. **(2 Points)**

b. Derive a relation between the average kinetic energy of the electrons and the Fermi energy. **(1 Points)**

c. The mass density is 10^{12}kg/m^3. Calculate the average kinetic energy of an electron, in MeV. (One MeV=1.6 x 10^{-13}J.) **(1 Points)**

d. The temperature is 10^9K. Calculate the average kinetic energy of the nuclei. **(1 Points)**

e. According to the virial theorem the internal energy of a system is approximately equal to its gravitational potential energy. For a sphere of uniform density, the gravitational potential energy is $3GM^2/5R$. Derive an expression for the mass of the white dwarf in terms of fundamental constants only. **(3 Points)**

f. Calculate the mass of the white dwarf in solar masses. (1 solar mass = 2 x 10^{30}kg). A white dwarf in which the electrons are relativistic is unstable with respect to collapse, so the quantity that you have calculated is approximately the maximum mass of a white dwarf, a quantity called the Chandrasekhar mass (1.4 solar masses). Does your numerical result look reasonable? Why or why not? **(2 Points)**

6.

a) spin degeneracy ↓

$$g(E) = g_s \frac{1}{(2\pi)^3} \int d^3x\, d^3k\; \delta(E_k - E)$$

$E_k = \hbar k c$

$dE = \hbar c\, dk$

$k^2 = \frac{E^2}{\hbar^2 c^2}$

$$= 2 \frac{V}{(2\pi)^3} 4\pi \int k^2 dk\; \delta(E_k - E)$$

$$= \frac{2V}{(2\pi)^3} \frac{4\pi}{(\hbar c)^3} \int_0^\infty E_k^2\, \delta(E_k - E_F)\, dE_k$$

$$g(E) = \frac{8\pi V}{(2\pi \hbar c)^3} E_F^2 = \frac{V}{\pi^2 \hbar^3 c^3} E_F^2 \checkmark$$

$$N = \int g(E)\, f(E)\, dE$$

At $T=0$ $f(E) = \Theta(E_F - E)$

$$= \frac{8\pi V}{(2\pi \hbar c)^3} \int_0^\infty E^2\, \Theta(E_F - E)\, dE$$

$$= \frac{8\pi V}{(2\pi \hbar c)^3} \int_0^{E_F} E^2\, dE$$

$$= \frac{1}{3} \frac{8\pi V}{(2\pi \hbar c)^3} E_F^3$$

$$\Rightarrow E_F^3 = 3N \frac{(2\pi \hbar c)^3}{8\pi V} = 3 \frac{N}{V} \pi^2 \hbar^3 c^3$$

$$\Rightarrow E_F = \left(3 \frac{N}{V} \pi^2\right)^{1/3} \hbar c$$

b) $\langle E \rangle = \int g(\epsilon) f(\epsilon)\, \epsilon\, d\epsilon$

$$= \frac{V}{\pi^2 (\hbar c)^3} \int E^3\, \Theta(E_F - E)\, dE$$

$$\langle T \rangle = \frac{\langle E \rangle}{V} = \frac{1}{4\pi^2 (\hbar c)^3} E_F^4$$

$$\Rightarrow \quad \frac{\langle E \rangle}{V} = \frac{1}{4\pi^2 (\hbar c)^3} \left(3 \frac{N}{V} \pi^2\right)^{4/3} \hbar c$$

$$= \frac{(3\pi^2)^{4/3}}{4 (\pi \hbar c)^2}\, n^{4/3}$$

c) $\rho = 10^{12}\ \mathrm{kg/m^3}$

* Consider equal amount of ^{12}C and ^{16}O

$$\rho_c = \frac{1}{2}\rho = \rho_o$$

* Convert the mass from 1 mol to 1 particle

$$m_c = \frac{12 \times 10^{-3}\ \mathrm{kg}}{N_A}, \quad m_o = \frac{16 \times 10^{-3}\ \mathrm{kg}}{N_A}$$ → Avogadro's #

gives you # ^{12}C & ^{16}O in m^{-3}

$$\rightsquigarrow\ n_c = \frac{\rho_c}{m_c} \frac{\mathrm{kg/m^3}}{\mathrm{kg}} = \frac{\frac{1}{2} \times 10^{12}}{12 \times 10^{-3}} N_A\ m^{-3} = \frac{10^{15}}{24} N_A\ m^{-3}$$

$$\rightsquigarrow\ n_o = \frac{10^{15}}{32} N_A\ m^{-3}$$

* each C has 6 electrons
* each O has 8 electrons

Thus, $n = \frac{N}{V} = 6 * n_C + 8 * n_O$

Thus, average kinetic energy of the nuclei is

$$\langle T \rangle = \frac{3\pi^2}{4(\pi\hbar c)^2} (6n_C + 8n_O)^{4/3} \frac{1}{1.6 \times 10^{-13}} \text{ MeV}$$

d) $\langle E \rangle = \langle T \rangle = \frac{f}{2} NKT$

e) $\langle T \rangle = \frac{3GM^2}{5R}$

$$\Rightarrow \frac{1}{4} \frac{(3\pi^2)^{4/3}}{(\pi\hbar c)^2} n^{4/3} = \frac{3GM^2}{5R}$$