

# Physics 5153 Classical Mechanics

## Principle of Virtual Work

### 1 Introduction

The first variational principle we encounter in mechanics is the principle of virtual work. It establishes the equilibrium condition of a mechanical system, and is fundamental for the later development of analytical mechanics (Lagrangian and Hamiltonian methods).

The concept of virtual work is centered on the idea of calculating the amount of work done on a system of particles through a virtual displacement. We will start by defining what we mean by a virtual displacement, then discuss virtual work, state the principle of virtual work, and consider an example of its use.

#### 1.1 Virtual Displacement

To define what we mean by a virtual displacement, let's consider a system composed of  $N$  particles, possibly subject to some set of constraints, defined by  $3N$  Cartesian coordinates ( $x_i$ ) relative to an inertial frame. Let's assume that at some instant of time the system undergoes infinitesimal displacements that are virtual in the sense that they occur without the passage of time (instantaneous), do not necessarily conform to the constraints. This change,  $\delta\vec{x}_i$ , in the configuration of the system is known as a virtual displacement.

In the usual case, a virtual displacement conforms to the instantaneous constraints, that is, moving constraints are assumed stopped during the displacement. For example, consider a system subject to  $n$  holonomic constraints

$$f_i(x_1, \dots, x_{3N}, t) = 0 \quad (1)$$

A total derivative corresponds to a infinitesimal displacement of the system and is give by

$$df_i = \sum_j \frac{\partial f_i}{\partial x_j} dx_j + \frac{\partial f_i}{\partial t} dt = 0 \quad (2)$$

notice that this gives both a spacial and a temporal displacement. In the case of a virtual displacement, we assume that the temporal displacement is zero, therefore the constraint changes by

$$\delta f_i = \sum_j \frac{\partial f_i}{\partial x_j} \delta x_j = 0 \quad (3)$$

It is important to note the difference, the displacement occurs in zero time.

One question that may be asked, are there any conditions under which a real and virtual displacement are the same? The answer can be seen by comparing Eqs. 2 and 3. If the constraint equation is scleronomic, a virtual displacement is the same as a real displacement. Therefore, in the general case virtual and real displacements are not the same, but in the scleronomic case they are.

Before concluding our discussion of virtual displacements, let's consider the nonholonomic case where the constraint is given in terms of derivatives. Assume  $n$  constraint equations on a system of  $3N$  degrees of freedom

$$\sum_i a_{ji} dx_i + a_{jt} dt = 0 \quad (4)$$

where  $j$  corresponds to the  $j^{\text{th}}$  constraint. Based on our definition of a virtual displacement consistent with the constraints, a virtual displacement for nonholonomic constraints is given by

$$\sum_i a_{ji} \delta x_i = 0 \quad (5)$$

This equation will become important later when we discuss calculating forces of constraint through the Lagrange multiplier method.

So far we have considered virtual displacements in terms of Cartesian coordinates. Virtual displacements in terms of generalized coordinates are also possible. Simply transform the Cartesian constraint equations to the generalized coordinates. The form of the constraint equation is given by

$$\sum_i a_{ji} dq_i + a_{jt} dt = 0 \quad (6)$$

where replacing the  $a$  with

$$\frac{\partial f_j}{\partial q_i} \quad \text{and} \quad \frac{\partial f_j}{\partial t} \quad (7)$$

gives the holonomic constraint. For a virtual displacement, the constraint equation becomes

$$\sum_i a_{ji} \delta q_i = 0 \quad (8)$$

Therefore the form is the same using any set of coordinates.

## 1.2 Virtual Work

Let's again consider a system of  $N$  particles with  $3N$  degrees of freedom whose configuration is given by the Cartesian coordinates  $x_1 \dots x_{3N}$ . In addition, suppose that the forces  $F_1 \dots F_{3N}$  are acting on the particles at the corresponding coordinates in a positive sense. The virtual work is given by

$$\delta W = \sum_i F_i \delta x_i = \sum_i \vec{\mathbf{F}}_i \cdot \delta \vec{\mathbf{r}}_i \quad (9)$$

The second equality implies that the virtual work is independent of coordinates used. The equation can be transformed as follows to any set of generalized coordinates

$$\delta W = \sum_j \left( \sum_i F_i \frac{\partial x_i}{\partial q_j} \right) \delta q_j \quad (10)$$

From this equation, we define the generalized force as

$$Q_j = \left( \sum_i F_i \frac{\partial x_i}{\partial q_j} \right) \Rightarrow \delta W = \sum_j Q_j \delta q_j \quad (11)$$

where we note that the generalized force does not have to have units of a force, just like the generalized coordinates do not have to have units of a length. But, the product of generalized force and coordinates has the units of work (energy).

In the expression for virtual work, the forces are assumed to remain constant throughout the virtual displacement. This is true even if the forces vary drastically over a infinitesimal displacement. A sudden change of force with position can occur in certain nonlinear systems.

Now assume that the system is subject to constraints. The force can be separated into applied forces  $\vec{\mathbf{F}}^a$  and constraint forces  $\vec{\mathbf{F}}^c$ . The virtual work of the constraint forces in terms of generalized coordinates is given by

$$\delta W_c = \sum_i Q_i^c \delta q_i \quad (12)$$

If the displacement is consistent with the constraint, the virtual work is zero since the force does not act in the direction of the force

$$\delta W_c = \sum_i Q_i^c \delta q_i = 0 \quad (13)$$

which is referred to as a workless constraint. These will be the type of constraint that we will deal with most often. If the constraints are workless, then the total virtual work on the system is given by the applied forces

$$\delta W = \sum_i Q_i^a \delta q_i \quad (14)$$

### 1.3 Principle of Virtual Work

One of the important applications of the idea of virtual work arises in the study of static equilibrium of mechanical system. Assume a scleronomic system of  $N$  particles. If the system is in static equilibrium, then Newton's laws for each of the  $N$  particles give

$$\vec{\mathbf{F}}_i^a + \vec{\mathbf{F}}_i^c = 0 \quad (15)$$

The virtual work for this system is given by

$$\delta W = \sum_i \vec{\mathbf{F}}_i^a \cdot \delta \vec{\mathbf{r}}_i + \vec{\mathbf{F}}_i^c \cdot \delta \vec{\mathbf{r}}_i = 0 \quad (16)$$

If we now assume that the constraints are workless, and the virtual displacements reversible (one can replace  $\delta \vec{\mathbf{r}}$  with  $-\delta \vec{\mathbf{r}}$ ), then the condition for static equilibrium is

$$\delta W = \sum_i \vec{\mathbf{F}}_i^a \cdot \delta \vec{\mathbf{r}}_i = 0 \quad \Rightarrow \quad \delta W = \sum_i Q_i^a \delta q_i = 0 \quad (17)$$

where the second equation is given using generalized coordinates. A very important point to note here is that unlike the Newtonian approach, we do not need to know what the constraint forces are. We only need to know the applied forces.

Now assume that the system is initially motionless, but not in equilibrium. Then one or more of the particles has a net applied force on it, and in accord with Newton's laws, it will start to move in the direction of the force. Since any motion must be compatible with the constraints, the

virtual displacements can be chosen to be in the direction of the actual motion at each point. In this case the virtual work is positive

$$\delta W = \sum_i \vec{\mathbf{F}}_i^a \cdot \delta \vec{\mathbf{r}}_i + \vec{\mathbf{F}}_i^c \cdot \delta \vec{\mathbf{r}}_i > 0 \quad (18)$$

Since the constraints are workless, the condition becomes

$$\delta W = \sum_i \vec{\mathbf{F}}_i^a \cdot \delta \vec{\mathbf{r}}_i > 0 \quad (19)$$

If the virtual displacements are reversed, then the virtual work is negative. None-the-less, if the system is not in equilibrium, one can find a set of virtual displacements that will result in the virtual work being nonzero.

These results can be summarized in the principle of virtual work: *The necessary and sufficient condition for the static equilibrium of an initially motionless scleronomic system that is subject to workless constraints is that zero virtual work be done by the applied forces in moving through an arbitrary virtual displacement satisfying the constraints.*

#### 1.4 Example

As a simple consider the system described in Fig. 1, where we want to determine the force  $F$  that will keep the system in equilibrium. If we use the Newtonian approach, we require 3 equations to solve the problem

$$\begin{aligned} \sum F_x = 0 & \quad N_1 - F = 0 \\ \sum F_y = 0 & \quad N_2 - 2mg = 0 \\ \sum \tau = 0 & \quad mgl \cos \theta - N_1 \ell \sin \theta = 0 \end{aligned} \quad (20)$$

From this point it is fairly straight forward to solve the problem. One finds  $F = mg \cot \theta$ .

Using the principle of virtual work, we set up the equation as follows

$$mg\delta y - F\delta x = 0 \quad (21)$$

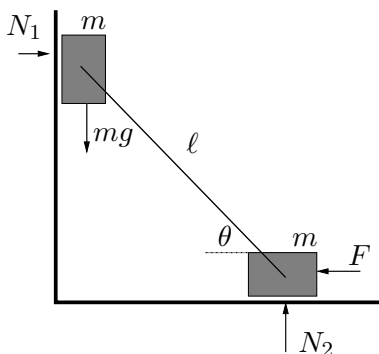


Figure 1: Two blocks on frictionless surfaces constrained by a rod to move together.

with the constraint between  $x$  and  $y$

$$\left. \begin{array}{l} x = \ell \cos \theta \\ y = \ell \sin \theta \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} \delta x = -\delta\theta \ell \sin \theta \\ \delta y = \delta\theta \ell \cos \theta \end{array} \right\} \Rightarrow \delta x \cos \theta - \delta y \sin \theta = 0 \quad (22)$$

since I have already assumed directions for  $\delta x$  and  $\delta y$  in Eq. 21, the sign here is dropped between the middle and final equations. Combining the two equations

$$(mg \cot \theta - F)\delta x = 0 \quad (23)$$

Since the displacement is arbitrary, and this equation must hold for all possible virtual displacements, the quantity inside the parenthesis must be zero

$$F = mg \cot \theta \quad (24)$$

the same as the Newtonian method.

The point of this example is not to show that one method is superior to the other, but that differences in the two methods. In the Newtonian method, we required the constraint force and a set of 3 equations to specify the problem. Using the method of virtual work we need only two equations, one describing the work done and the second describing the constraints. We don't need the constraint forces.