Physics 5153 Classical Mechanics
Generalized Coordinates and Constraints

1 Introduction

Dynamics is the study of the motion of interacting objects. It describes the motion in terms of a set of postulates. In our case we will deal with classical dynamics, where we restrict ourselves to system where quantum and relativistic effects are negligible.

This lecture introduces some basic concepts of classical dynamics, and represents the beginning of our study of analytical mechanics. We will start with a brief discussion of Newtonian mechanics and what we expect to gain from the analytic approach. This will be followed by a discussion of generalized coordinates, and then a discussion of constraints.

1.1 Analytical and Vectorial Mechanics

The analytical form of mechanics differs considerably from the vectorial form. The analytical form of mechnics is based on the work function and kinetic energy, while the vectorial form is based directly on Newton's three laws, in particular the second law. The following list briefly describes those differences, these will become clearer as the semester progresses:

1. The vectorial form requires that each particle be isolated with the forces acting on it be given with each particle forming a separate equation. The analytic method treats the system as a whole with one equation describing the system in terms of the work (potential energy).

2. If there are strong forces that maintain a fixed relation between the coordinates of the particles in the system, and an empirical relation that describes it, the vectorial method has to consider the forces that are required to maintain it. The analytic method uses the given relation between the coordinates, this then corresponds to the imposed constraints on the particles.

3. The equations of motion in the analytic method are derived from a single principle that implicitly includes all the equations. The principle requires that the "action" (quantity that describes the system) be minimized. Since the minimization is independent of the coordinate system used, this allows one to select a set of coordinates that fit the problem.

1.2 Generalized Coordinates

An important characteristic of any mechanical system is the number of degrees of freedom. The number of degrees of freedom is the number of coordinates needed to specify the location of the objects. Therefore, if there are N free objects, there are $3N$ degrees of freedom. But if there are constraints on the objects, then each constraint removes one degree of freedom. The total number of degrees of freedom for a system of N objects and n constraints is $3N - n$.

As an example, consider 3 free objects. This system has a total of 9 degrees of freedom. If we constrain the separation between the three to be fixed, we loose three degrees of freedom and therefore our system has only $9 - 3 = 6$ degrees of freedom. These 6 degrees of freedom could be selected using any convenient set, for example the 3 coordinates of the center of mass plus the

3 Euler angles. Note, the number of degrees of freedom is independent of the set of coordinates selected.

From the above discussion, we are free to select the set of coordinates used to describe a given system as long as the number of coordinates minus the number of constraints gives the number of degrees of freedom for the system. Note that the number of coordinates and number of constraints does not have to be the same for all possible choices, as illustrated in the 3 mass example 9 coordiantes to 6 coordinates 3 constraints to zero constraints, but the difference must be constant.

To describe a system, we can use any set of parameters that unambiguously represent it. These parameters do not need to have dimensions of length. They are referred to as generalized coordinates. The best choice of generalized coordinates for any given system is one that takes advantage of any symmetries it may have and one where the coordinates are independent. Coordinates that are tied to the symmetry of the problem, lead to conserved momenta. If the coordinates can be varied independently without violating the constraints, their number is equal to the number of degrees of freedom.

To simplify most problems, one starts by writing the dynamical equations in rectangular coordinates. Then one transforms to the generalized coordinates. The transformation equations between the two sets is given formally by the following transformation equations

$$\begin{aligned} x_1 &= x_1(q_1, q_2, \ldots, q_{3N}) \quad (1) \\ y_1 &= y_1(q_1, q_2, \ldots, q_{3N}) \\ z_1 &= z_1(q_1, q_2, \ldots, q_{3N}) \\ &\vdots \\ x_{3N} &= x_{3N}(q_1, q_2, \ldots, q_{3N}) \\ y_{3N} &= y_{3N}(q_1, q_2, \ldots, q_{3N}) \\ z_{3N} &= z_{3N}(q_1, q_2, \ldots, q_{3N}) \quad (2) \end{aligned}$$

The constraints are selected at a fixed time, and they are required to be continuous, differentiable, and single valued.

The q_i describe the configuration of the system. If they are independent of each other, they form a $3N$ dimensional space called configuration space. The properties of the configuration space are given by the differential displacements of the sets of coordinates

$$dx_i = \sum_j \frac{\partial x_i}{\partial q_j} dq_j \quad (3)$$

The partial derivatives define the scale between the two sets and are referred to as the metric of the space. The differential volume is given by the product of the Jacobian and the product of the differential length elements

$$dx_1 dx_2 \cdots dx_{3N} = \left| \frac{\partial(x_1 x_2 \cdots x_{3N})}{\partial(q_1 q_2 \cdots q_{3N})} \right| dq_1 dq_2 \cdots d_{3N} \quad (4)$$

where

$$\left| \frac{\partial(x_1 x_2 \cdots x_{3N})}{\partial(q_1 q_2 \cdots q_{3N})} \right| = \begin{vmatrix} \frac{\partial x_1}{\partial q_1} & \frac{\partial x_1}{\partial q_2} & \cdots \\ \frac{\partial x_2}{\partial q_1} & \frac{\partial x_2}{\partial q_2} & \cdots \\ \vdots & \vdots & \ddots \end{vmatrix} \quad (5)$$

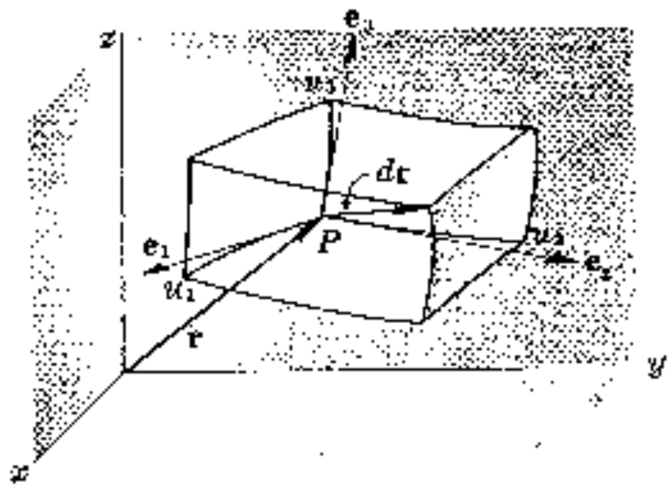

Figure 1: Volume element in curvilinear coordinates.

This can be seen most clearly in three dimensions. The three differentials are the lengths of the three sides of a parallelepiped (see Fig. 1). The volume of the parallelepiped is given by

$$V = \vec{\mathbf{u}}_1 \cdot \vec{\mathbf{u}}_2 \times \vec{\mathbf{u}}_3 = \begin{vmatrix} u_{x1} & 0 & 0 \\ 0 & u_{y2} & 0 \\ 0 & 0 & u_{z3} \end{vmatrix} = u_{x1} u_{y2} u_{z3}\, qRarrow dV = dx\, dy\, dz = h_1 dq_1 h_2 dq_2 h_3 dq_3 \quad \text{where} \quad h_i = \left| \frac{\partial \vec{\mathbf{r}}}{\partial q_i} \right| \tag{6}$$

If the Jacobian is zero, this implies that rectangular coordinates map over into a region of zero volume in the configuration space, there the is no mapping between the two sets. Therefore, for a transformation to exists, the Jacobian must be nonzero. In other words a finite volume in one space must transform into a finite volume in the other space so that points can be transformed from one space to the other. In this case the points map one to one onto the new space. This type of transformations is called a point transformation. These will be the transformations used in Lagrangian dynamics. For Hamiltonian dynamics, we will have more powerful transformations avalible to us.

Let's consider a simple example of a transformation. Consider a particle traveling along a fixed circular path. Its position can be written in Cartesian coordinates, but a more logical set would be polar coordinates. The transformation equations are given by

$$x = r\cos\theta \qquad y = r\sin\theta \tag{7}$$

the Jacobian for this system is

$$\begin{vmatrix} -r\sin\theta & \cos\theta \\ r\cos\theta & \sin\theta \end{vmatrix} = -r \tag{8}$$

so for $r > 0$ this transformation works, for $r = 0$ this transformation will not work, which should be obvious because then θ is not defined.

1.3 Constraints

Many of the problems that we have to solve have constraints of one type or another. Take the example of the double pendulum. The lower mass has both a motion that is independent of the upper mass, and one that is tied to it. Using the Newtonian approach to mechanics, we treat the constraints as either normal forces, tensions, stresses, ... Based on the fact that the constraints are treated as forces, Newton's second law can be written to take into account that there are two types of forces, the applied forces, gravity, electric, magnetic, contact (*pushing or pulling*), and the constraint forces, those forces required for the object to follow the prescribed path or stay within the prescribed boundaries

$$m\vec{\mathbf{a}} = \vec{\mathbf{F}}_a + \vec{\mathbf{F}}_c \tag{9}$$

where $\vec{\mathbf{F}}_a$ are the applied forces and $\vec{\mathbf{F}}_c$ are the forces of constraint.

Since analytical mechanics is based on work (energies), there is no energy associated directly with the constraint force. Therefore, the constraints are introduced as relations between the coordinates. Since there are a variety of different types of constraints, we will examine some of the possibilities and how to incorporate them into the Lagrangian or Hamiltonian. The simplest constraints are those that can be put into the following form

$$g_j(q_i) = 0 \tag{10}$$

where g_j can be a function of all the generalized coordinates. Constraints of this form are called holonomic. Holonomic constraints can be either time independent (scleronomic) or time dependent (rheonomic). A few examples of Holonomic constraints are

Block sliding on a wedge	$y - ax - b = 0$
Block sliding on an incline plane angle varies	$y - x\tan(\omega t) = 0$
Mass attached to a rod (pendulum)	$r - \ell = 0$
Bead on a rigid wire	$f(q_i) = 0$

All constriants that are not given by $g_j(q_i) = 0$ are called non-holonomic. One example is a particle confined to a box, the constraint is given by $f(\vec{\mathbf{r}}) > 0$. Another far more useful example is one that is given in terms of derivatives

$$\sum_i a_{ji}(q_i)dq_i + a_{jt}dt = 0 \tag{11}$$

where the sum is over the total number of generalized coordinates. There are two cases that we have to consider for this type of constraint. The first is if the sum can be written as a total differential

$$\sum_i a_i(q_i)dq_i \quad \text{where} \quad a_i = \frac{\partial g}{\partial q_i} \tag{12}$$

where g is the constraint. This equation can be integrated and therefore put into holonomic form. Note that the condition of integrability is given by

$$\frac{\partial a_j}{\partial q_i} = \frac{\partial^2 g}{\partial q_i \partial q_j} = \frac{\partial a_i}{\partial q_j} \tag{13}$$

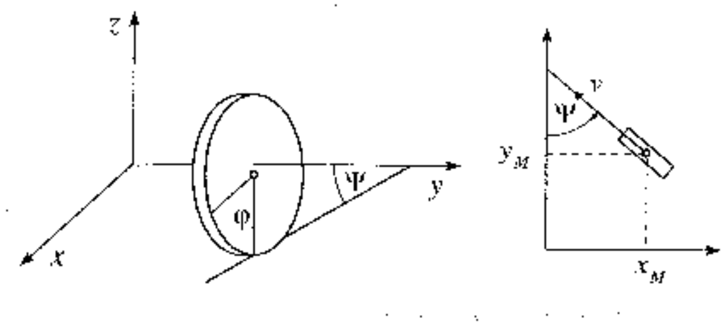

Figure 2: Wheel constrained to roll on a plane surface.

if this condition is not satisfied, then the constraint is non-holonomic. As an example, consider a wheel rolling on a plane (see Fig. 2). The condition for rolling is given by

$$v = a\dot{\phi} \tag{14}$$

where a is the radius of the wheeel. The x and y components of the center of mass velocity are given by

$$\dot{x}_m = -v \sin\psi \qquad \dot{y}_m = v\cos\psi \tag{15}$$

Combining these equations together, we arrive at

$$\begin{aligned} dx_m + a\sin(\psi)d\phi = 0 \\ dy_m - a\cos(\psi)d\phi = 0 \end{aligned} \tag{16}$$

if ψ is a constant, then this equation is integrable, if it is only known after we solve the equations of motion, then this is a non-holonomic constraint.