Quantum Geometry

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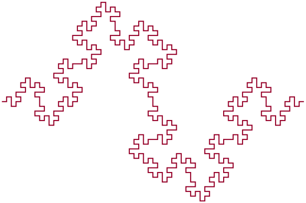

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Electronic Band Structure

- In quantum mechanics, eigenvalues and eigenstates fully describe the physical behavior of a system.
- Historically, the eigenvalues are primarily emphasized.
  - These include energy, momentum, spin, etc. — all observable quantities!
  - In solids, electronic band structure describes the allowed range of energy levels that electrons may occupy.

Quantum Geometric Tensor

However, there is also a rich quantum geometry of the (projective) Hilbert space itself.

- Admits definition of distances between quantum states.

The geometry of the Hilbert space is captured in the quantum geometric tensor:

$$T_{ij}(u_\lambda) \equiv \left\langle \partial_i u_\lambda \,\middle|\, \left(1 - |u_\lambda\rangle\langle u_\lambda|\right) \,\middle|\, \partial_j u_\lambda \right\rangle$$

(where $\partial_i = \partial/\partial(k_i)$ with $i = x, y$ and u_λ is a wave function parametrized by some quantity λ).

Berry Phase

The complex phase of a quantum state is usually thought of as having no physical significance. However, the relative phase between states can have crucial physical consequences. The

Berry phase describes how a (global) phase accumulates as a complex vector is carries about a closed loop in a complex vector space:

$$\langle \psi_{initial} \mid \psi_{final} \rangle = \exp\left(i\phi\right)$$

(in a "parallel-transport gauge").

Berry Phase

In the continuum limit, the Berry phase can be expressed as:

$$\phi = \oint \langle u_\lambda \mid i\partial_\lambda u_\lambda \rangle \mathrm{d}\lambda = \oint A(\lambda) \mathrm{d}\lambda$$

The integrand is called the Berry connection:

$$A(\lambda) = \langle u_\lambda \mid i\partial_\lambda u_\lambda \rangle$$

Berry Curvature

The Berry curvature is simply the Berry phase per unit area in parameter space.

$$\Phi_S = \int_S \Omega(\lambda) \mathrm{d}S = \oint_P A \cdot \mathrm{d}\lambda = \phi_P$$

The imaginary part of T_{ij} is a (real) antisymmetric second-rank tensor; this is exactly half the Berry curvature:

$$T_{ij} \equiv g_{ij} + \frac{i}{2}\Omega_{ij}$$

Quantum Metric

The real part of the quantum metric tensor T_{ij} is called the quantum metric.

$$T_{ij} \equiv g_{ij} + \frac{i}{2}\Omega_{ij}$$

- I'm interested in material properties which are affected by the quantum metric.

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