CHAPTER VIII

ASYMPTOTIC EXPANSIONS AND SUMMABLE SERIES

8·1. *Simple example of an asymptotic expansion.*

Consider the function $f(x) = \int_x^\infty t^{-1} e^{x-t}\, dt$, where x is real and positive, and the path of integration is the real axis.

By repeated integrations by parts, we obtain

$$f(x) = \frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \ldots + \frac{(-)^{n-1}(n-1)!}{x^n} + (-)^n n! \int_x^\infty \frac{e^{x-t}\, dt}{t^{n+1}}.$$

In connexion with the function $f(x)$, we therefore consider the expression

$$u_{n-1} = \frac{(-)^{n-1}(n-1)!}{x^n},$$

and we shall write

$$\sum_{m=0}^{n} u_m = \frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \ldots + \frac{(-)^n n!}{x^{n+1}} = S_n(x).$$

Then we have $|u_m/u_{m-1}| = mx^{-1} \to \infty$ as $m \to \infty$. *The series Σu_m is therefore divergent for all values of x.* In spite of this, however, the series can be used for the calculation of $f(x)$; this can be seen in the following way.

Take any fixed value for the number n, and calculate the value of S_n. We have

$$f(x) - S_n(x) = (-)^{n+1}(n+1)! \int_x^\infty \frac{e^{x-t}\, dt}{t^{n+2}},$$

and therefore, since $e^{x-t} \leqslant 1$,

$$|f(x) - S_n(x)| = (n+1)! \int_x^\infty \frac{e^{x-t}\, dt}{t^{n+2}} < (n+1)! \int_x^\infty \frac{dt}{t^{n+2}} = \frac{n!}{x^{n+1}}.$$

For values of x which are sufficiently large, the right-hand member of this equation is very small. Thus, if we take $x \geqslant 2n$, we have

$$|f(x) - S_n(x)| < \frac{1}{2^{n+1} n^2},$$

which for large values of n is very small. It follows therefore that *the value of the function $f(x)$ can be calculated with great accuracy for large values of x, by taking the sum of a suitable number of terms of the series Σu_m.*

Taking even fairly small values of x and n

$$S_5(10) = 0{\cdot}09152, \text{ and } 0 < f(10) - S_5(10) < 0{\cdot}00012.$$

The series is on this account said to be an *asymptotic expansion* of the function $f(x)$. The precise definition of an asymptotic expansion will now be given.

8·2. *Definition of an asymptotic expansion.*

A divergent series

$$A_0 + \frac{A_1}{z} + \frac{A_2}{z^2} + \ldots + \frac{A_n}{z^n} + \ldots,$$

in which the sum of the first $(n+1)$ terms is $S_n(z)$, is said to be an *asymptotic expansion* of a function $f(z)$ for a given range of values of $\arg z$, if the expression $R_n(z) = z^n\{f(z) - S_n(z)\}$ satisfies the condition

$$\lim_{|z|\to\infty} R_n(z) = 0 \quad (n \text{ fixed}),$$

even though

$$\lim_{n\to\infty} |R_n(z)| = \infty \quad (z \text{ fixed}).$$

When this is the case, we can make

$$|z^n\{f(z) - S_n(z)\}| < \epsilon,$$

where ϵ is arbitrarily small, by taking $|z|$ sufficiently large.

We denote the fact that the series is the asymptotic expansion of $f(z)$ by writing

$$f(z) \sim \sum_{n=0}^{\infty} A_n z^{-n}.$$

The definition which has just been given is due to Poincaré*. Special asymptotic expansions had, however, been discovered and used in the eighteenth century by Stirling, Maclaurin and Euler. Asymptotic expansions are of great importance in the theory of Linear Differential Equations, and in Dynamical Astronomy; some applications will be given in subsequent chapters of the present work.

The example discussed in § 8·1 clearly satisfies the definition just given: for, when x is positive, $|x^n\{f(x) - S_n(x)\}| < n!\,x^{-1} \to 0$ as $x \to \infty$.

For the sake of simplicity, in this chapter we shall for the most part consider asymptotic expansions only in connexion with real positive values of the argument. The theory for complex values of the argument may be discussed by an extension of the analysis.

8·21. *Another example of an asymptotic expansion.*

As a second example, consider the function $f(x)$, represented by the series

$$f(x) = \sum_{k=1}^{\infty} \frac{c^k}{x+k},$$

where $x > 0$ and $0 < c < 1$.

* *Acta Mathematica*, VIII. (1886), pp. 295–344.

The ratio of the kth term of this series to the $(k-1)$th is less than c, and consequently the series converges for all positive values of x. We shall confine our attention to positive values of x. We have, when $x>k$,

$$\frac{1}{x+k}=\frac{1}{x}-\frac{k}{x^2}+\frac{k^2}{x^3}-\frac{k^3}{x^4}+\frac{k^4}{x^5}-\dots.$$

If, therefore, it were allowable* to expand each fraction $\frac{1}{x+k}$ in this way, and to rearrange the series for $f(x)$ in descending powers of x, we should obtain the formal series

$$\frac{A_1}{x}+\frac{A_2}{x^2}+\dots+\frac{A_n}{x^n}+\dots,$$

where
$$A_n=(-)^{n-1}\sum_{k=1}^{\infty}k^{n-1}c^k.$$

But this procedure is not legitimate, and in fact $\sum_{n=1}^{\infty}A_n x^{-n}$ diverges. We can, however, shew that it is an asymptotic expansion of $f(x)$.

For let
$$S_n(x)=\frac{A_1}{x}+\frac{A_2}{x^2}+\dots+\frac{A_n}{x^{n+1}}.$$

Then
$$S_n(x)=\sum_{k=1}^{\infty}\left(\frac{c^k}{x}-\frac{kc^k}{x^2}+\frac{k^2c^k}{x^3}+\dots+\frac{(-)^n k^n c^k}{x^{n+1}}\right)$$
$$=\sum_{k=1}^{\infty}\left\{1-\left(-\frac{k}{x}\right)^{n+1}\right\}\frac{c^k}{x+k};$$

so that
$$|f(x)-S_n(x)|=\left|\sum_{k=1}^{\infty}\left(-\frac{k}{x}\right)^{n+1}\frac{c^k}{x+k}\right|<x^{-n-2}\sum_{k=1}^{\infty}k^n c^k.$$

Now $\sum_{k=1}^{\infty}k^n c^k$ converges for any given value of n and is equal to C_n, say; and hence $|f(x)-S_n(x)|<C_n x^{-n-2}$.

Consequently
$$f(x)\sim\sum_{n=1}^{\infty}A_n x^{-n}.$$

Example. If $f(x)=\int_x^{\infty}e^{x^2-t^2}\,dt$, where x is positive and the path of integration is the real axis, prove that

$$f(x)\sim\frac{1}{2x}-\frac{1}{2^2x^3}+\frac{1.3}{2^3x^5}-\frac{1.3.5}{2^4x^7}+\dots.$$

[In fact, it was shewn by Stokes in 1857 that

$$\int_0^x e^{x^2-t^2}\,dt\sim\pm\tfrac{1}{2}e^{x^2}\sqrt{\pi}-\left(\frac{1}{2x}-\frac{1}{2^2x^3}+\frac{1.3}{2^3x^5}-\frac{1.3.5}{2^4x^7}+\dots\right);$$

the upper or lower sign is to be taken according as $-\frac{1}{2}\pi<\arg x<\frac{1}{2}\pi$ or $\frac{1}{2}\pi<\arg x<\frac{3}{2}\pi$.]

8·3. *Multiplication of asymptotic expansions.*

We shall now shew that two asymptotic expansions, valid for a common range of values of $\arg z$, can be multiplied together in the same way as ordinary series, the result being a new asymptotic expansion.

For let
$$f(z)\sim\sum_{m=0}^{\infty}A_m z^{-m},\quad \phi(z)\sim\sum_{m=0}^{\infty}B_m z^{-m},$$

* It is not allowable, since $k>x$ for all terms of the series after some definite term.

and let $S_n(z)$ and $T_n(z)$ be the sums of their first $(n+1)$ terms; so that, n being fixed,

$$f(z) - S_n(z) = o(z^{-n}), \quad \phi(z) - T_n(z) = o(z^{-n}).$$

Then, if $C_m = A_0 B_m + A_1 B_{m-1} + \ldots + A_m B_0$, it is obvious that*

$$S_n(z)\, T_n(z) = \sum_{m=0}^{n} C_m z^{-m} + o(z^{-n}).$$

But
$$\begin{aligned} f(z)\,\phi(z) &= \{S_n(z) + o(z^{-n})\}\{T_n(z) + o(z^{-n})\} \\ &= S_n(z)\, T_n(z) + o(z^{-n}) \\ &= \sum_{m=0}^{n} C_m z^{-m} + o(z^{-n}). \end{aligned}$$

This result being true for *any* fixed value of n, we see that

$$f(z)\,\phi(z) \sim \sum_{m=0}^{\infty} C_m z^{-m}.$$

8·31. *Integration of asymptotic expansions.*

We shall now shew that it is permissible to integrate an asymptotic expansion term by term, the resulting series being the asymptotic expansion of the integral of the function represented by the original series.

For let $f(x) \sim \sum_{m=2}^{\infty} A_m x^{-m}$, and let $S_n(x) = \sum_{m=2}^{n} A_m x^{-m}$.

Then, given any positive number ϵ, we can find x_0 such that

$$|f(x) - S_n(x)| < \epsilon\, |x|^{-n} \text{ when } x > x_0,$$

and therefore

$$\left| \int_x^{\infty} f(x)\,dx - \int_x^{\infty} S_n(x)\,dx \right| \leqslant \int_x^{\infty} |f(x) - S_n(x)|\,dx$$
$$< \frac{\epsilon}{(n-1)\,x^{n-1}}.$$

But
$$\int_x^{\infty} S_n(x)\,dx = \frac{A_2}{x} + \frac{A_3}{2x^2} + \ldots + \frac{A_n}{(n-1)\,x^{n-1}},$$

and therefore
$$\int_x^{\infty} f(x)\,dx \sim \sum_{m=2}^{\infty} \frac{A_m}{(m-1)\,x^{m-1}}.$$

On the other hand, it is not in general permissible† to differentiate an asymptotic expansion; this may be seen by considering $e^{-x}\sin(e^x)$.

8·32. *Uniqueness of an asymptotic expansion.*

A question naturally suggests itself, as to whether a given series can be

* See § 2·11; we use $o(z^{-n})$ to denote *any* function $\psi(z)$ such that $z^n\psi(z) \to 0$ as $|z| \to \infty$.

† For a theorem concerning differentiation of asymptotic expansions representing analytic functions, see Ritt, *Bull. American Math. Soc.* xxiv. (1918), pp. 225–227.

the asymptotic expansion of several distinct functions. The answer to this is in the affirmative. To shew this, we first observe that there are functions $L(x)$ which are represented asymptotically by a series all of whose terms are zero, i.e. functions such that $\lim_{x\to\infty} x^n L(x) = 0$ for every fixed value of n. The function e^{-x} is such a function when x is positive. The asymptotic expansion* of a function $J(x)$ is therefore also the asymptotic expansion of

$$J(x) + L(x).$$

On the other hand, a function cannot be represented by more than one distinct asymptotic expansion over the whole of a given range of values of z; for, if

$$f(z) \sim \sum_{m=0}^{\infty} A_m z^{-m}, \quad f(z) \sim \sum_{m=0}^{\infty} B_m z^{-m},$$

then
$$\lim_{z\to\infty} z^n \left(A_0 + \frac{A_1}{z} + \ldots + \frac{A_n}{z^n} - B_0 - \frac{B_1}{z} - \ldots - \frac{B_n}{z^n}\right) = 0,$$

which can only be if
$$A_0 = B_0; \; A_1 = B_1, \; \ldots.$$

Important examples of asymptotic expansions will be discussed later, in connexion with the Gamma-function (Chapter XII) and Bessel functions (Chapter XVII).

8·4. *Methods of 'summing' series.*

We have seen that it is possible to obtain a development of the form

$$f(x) = \sum_{m=0}^{n} A_m x^{-m} + R_n(x),$$

where $R_n(x) \to \infty$ as $n \to \infty$, and the series $\sum_{m=0}^{\infty} A_m x^{-m}$ does not converge.

We now consider what meaning, if any, can be attached to the 'sum' of a non-convergent series. That is to say, given the numbers $a_0, a_1, a_2, \ldots$, we wish to formulate definite rules by which we can obtain from them a number S such that $S = \sum_{n=0}^{\infty} a_n$ if $\sum_{n=0}^{\infty} a_n$ converges, and such that S exists when this series does not converge.

8·41. *Borel's† method of summation.*

We have seen (§ 7·81) that

$$\sum_{n=0}^{\infty} a_n z^n = \int_0^{\infty} e^{-t} \phi(tz)\, dt,$$

where $\phi(tz) = \sum_{n=0}^{\infty} \frac{a_n t^n z^n}{n!}$, the equation certainly being true inside the circle of convergence of $\sum_{n=0}^{\infty} a_n z^n$. If the integral exists at points z outside this circle, we define the 'Borel sum' of $\sum_{n=0}^{\infty} a_n z^n$ to mean the integral.

* It has been shewn that when the coefficients in the expansion satisfy certain inequalities, there is only one *analytic* function with that asymptotic expansion. See *Phil. Trans.* 213, A, (1911), pp. 279–313.

† Borel, *Leçons sur les Séries Divergentes* (1901), pp. 97–115.

Thus, whenever $R(z) < 1$, the 'Borel sum' of the series $\sum_{n=0}^{\infty} z^n$ is

$$\int_0^\infty e^{-t} e^{tz} dt = (1-z)^{-1}.$$

If the 'Borel sum' exists we say that the series is 'summable (B).'

8·42. *Euler's* method of summation.*

A method, practically due to Euler, is suggested by the theorem of § 3·71; the 'sum' of $\sum_{n=0}^{\infty} a_n$ may be defined as $\lim_{x \to 1-0} \sum_{n=0}^{\infty} a_n x^n$, when this limit exists.

Thus the 'sum' of the series $1 - 1 + 1 - 1 + \ldots$ would be

$$\lim_{x \to 1-0} (1 - x + x^2 - \ldots) = \lim_{x \to 1-0} (1+x)^{-1} = \tfrac{1}{2}.$$

8·43. *Cesàro's† method of summation.*

Let $s_n = a_1 + a_2 + \ldots + a_n$; then *if* $S = \lim_{n \to \infty} \frac{1}{n}(s_1 + s_2 + \ldots + s_n)$ *exists*, we say that $\sum_{n=1}^{\infty} a_n$ is '*summable* $(C1)$,' and that its sum $(C1)$ is S. It is necessary to establish the 'condition of consistency‡,' namely that $S = \sum_{n=1}^{\infty} a_n$ when this series is convergent.

To obtain the required result, let $\sum_{m=1}^{\infty} a_m = s$, $\sum_{m=1}^{n} s_m = nS_n$; then we have to prove that $S_n \to s$.

Given ϵ, we can choose n such that $\left| \sum_{m=n+1}^{n+p} a_m \right| < \epsilon$ for all values of p, and so $|s - s_n| \leqslant \epsilon$.

Then, if $\nu > n$, we have

$$S_\nu = a_1 + a_2\left(1 - \frac{1}{\nu}\right) + \ldots + a_n\left(1 - \frac{n-1}{\nu}\right) + a_{n+1}\left(1 - \frac{n}{\nu}\right) + \ldots + a_\nu\left(1 - \frac{\nu-1}{\nu}\right).$$

Since $1, 1 - \nu^{-1}, 1 - 2\nu^{-1}, \ldots$ is a positive decreasing sequence, it follows from Abel's inequality (§ 2·301) that

$$\left| a_{n+1}\left(1 - \frac{n}{\nu}\right) + a_{n+2}\left(1 - \frac{n+1}{\nu}\right) + \ldots + a_\nu\left(1 - \frac{\nu-1}{\nu}\right) \right| < \left(1 - \frac{n}{\nu}\right)\epsilon.$$

Therefore

$$\left| S_\nu - \left\{ a_1 + a_2\left(1 - \frac{1}{\nu}\right) + \ldots + a_n\left(1 - \frac{n-1}{\nu}\right) \right\} \right| < \left(1 - \frac{n}{\nu}\right)\epsilon.$$

* *Instit. Calc. Diff.* (1755). See Borel, *loc. cit.* Introduction.

† *Bulletin des Sciences Math.* (2), XIV. (1890), p. 114.

‡ See the end of § 8·4.

Making $\nu \to \infty$, we see that, if S be any one of the limit points (§ 2·21) of S_ν, then

$$\left| S - \sum_{m=1}^{n} a_m \right| \leqslant \epsilon.$$

Therefore, since $|s - s_n| \leqslant \epsilon$, we have

$$|S - s| \leqslant 2\epsilon.$$

This inequality being true for *every* positive value of ϵ we infer, as in § 2·21, that $S = s$; that is to say S_ν has the unique limit s; this is the theorem which had to be proved.

Example 1. Frame a definition of 'uniform summability (C 1) of a series of variable terms.'

Example 2. If $b_{n,\nu} \geqslant b_{n+1,\nu} \geqslant 0$ when $n < \nu$, and if, when n is *fixed*, $\lim_{\nu\to\infty} b_{n,\nu} = 1$, and if $\sum_{m=1}^{\infty} a_m = s$, then $\lim_{\nu\to\infty} \left\{ \sum_{n=1}^{\nu} a_n b_{n,\nu} \right\} = S$.

8·431. *Cesàro's general method of summation.*

A series $\sum_{n=0}^{\infty} a_n$ is said to be 'summable (Cr)' if $\lim_{\nu\to\infty} \sum_{n=0}^{\nu} a_n b_{n,\nu}$ exists, where

$$b_{0,\nu} = 1,\ b_{n,\nu} = \left\{ \left(1 + \frac{r}{\nu+1-n}\right)\left(1 + \frac{r}{\nu+2-n}\right)\cdots\left(1 + \frac{r}{\nu-1}\right) \right\}^{-1}.$$

It follows from § 8·43 example 2 that the 'condition of consistency' is satisfied; in fact it can be proved* that if a series is summable (Cr') it is also summable (Cr) when $r > r'$; the condition of consistency is the particular case of this result when $r = 0$.

8·44. *The method of summation of Riesz*†.

A more extended method of 'summing' a series than the preceding is by means of

$$\lim_{\nu\to\infty} \sum_{n=1}^{\nu} \left(1 - \frac{\lambda_n}{\lambda_\nu}\right)^r a_n,$$

in which λ_n is any real function of n which tends to infinity with n. A series for which this limit exists is said to be 'summable (Rr) with sum-function λ_n.'

8·5. HARDY'S‡ CONVERGENCE THEOREM.

Let $\sum_{n=1}^{\infty} a_n$ be a series which is summable (C 1). Then if

$$a_n = O(1/n),$$

the series $\sum_{n=1}^{\infty} a_n$ converges.

* Bromwich, *Infinite Series*, § 122.

† *Comptes Rendus*, CXLIX. (1910), pp. 18–21.

‡ *Proc. London Math. Soc.* (2), VIII. (1910), pp. 302–304. For the proof here given, we are indebted to Mr Littlewood.

Let $s_n = a_1 + a_2 + \dots + a_n$; then since $\sum_{n=1}^{\infty} a_n$ is summable (C 1), we have

$$s_1 + s_2 + \dots + s_n = n\{s + o(1)\},$$

where s is the sum (C 1) of $\sum_{n=1}^{\infty} a_n$.

Let $$s_m - s = t_m, \quad (m = 1, 2, \dots n),$$

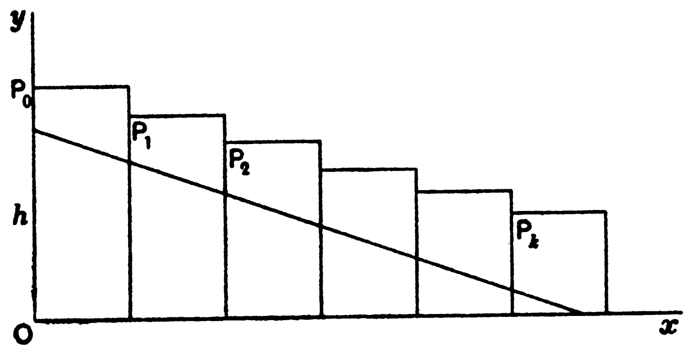
and let $$t_1 + t_2 + \dots + t_n = \sigma_n.$$

With this notation, it is sufficient to shew that, if $|a_n| < Kn^{-1}$, where K is independent of n, and if $\sigma_n = n \,.\, o(1)$, then $t_n \to 0$ as $n \to \infty$.

Suppose first that $a_1, a_2, \dots$ are real. Then, if t_n does not tend to zero, there is some positive number h such that there are an unlimited number of the numbers t_n which satisfy *either* (i) $t_n > h$ *or* (ii) $t_n < -h$. We shall shew that either of these hypotheses implies a contradiction. Take the former*, and choose n so that $t_n > h$.

Then, when $r = 0, 1, 2, \dots$,

$$|a_{n+r}| < K/n.$$



Now plot the points P_r whose coordinates are (r, t_{n+r}) in a Cartesian diagram. Since $t_{n+r+1} - t_{n+r} = a_{n+r+1}$, the slope of the line $P_r P_{r+1}$ is less than $\theta = \text{arc tan}\,(K/n)$.

Therefore the points $P_0, P_1, P_2, \dots$ lie above the line $y = h - x \tan\theta$. Let P_k be the last of the points $P_0, P_1, \dots$ which lie on the left of $x = h \cot\theta$, so that $k \leqslant h \cot\theta$.

Draw rectangles as shewn in the figure. The area of these rectangles exceeds the area of the triangle bounded by $y = h - x\tan\theta$ and the axes; that is to say

$$\sigma_{n+k} - \sigma_{n-1} = t_n + t_{n+1} + \dots + t_{n+k}$$
$$> \tfrac{1}{2} h^2 \cot\theta = \tfrac{1}{2} h^2 K^{-1} n.$$

* The reader will see that the latter hypothesis involves a contradiction by using arguments of a precisely similar character to those which will be employed in dealing with the former hypothesis.

But
$$\begin{aligned}|\sigma_{n+k}-\sigma_{n-1}| &\leqslant |\sigma_{n+k}|+|\sigma_{n-1}|\\ &=(n+k)\,.\,o(1)+(n-1)\,.\,o(1)\\ &=n\,.\,o(1),\end{aligned}$$

since $k \leqslant hnK^{-1}$, and h, K are independent of n.

Therefore, for a set of values of n tending to infinity,

$$\tfrac{1}{2}h^2K^{-1}n < n\,.\,o(1),$$

which is impossible since $\frac{1}{2}h^2K^{-1}$ is *not* $o(1)$ as $n \to \infty$.

This is the contradiction obtained on the hypothesis that $\overline{\lim}\, t_n \geqslant h > 0$; therefore $\overline{\lim}\, t_n \leqslant 0$. Similarly, by taking the corresponding case in which $t_n \leqslant -h$, we arrive at the result $\underline{\lim}\, t_n \geqslant 0$. Therefore since $\overline{\lim}\, t_n \geqslant \underline{\lim}\, t_n$, we have

$$\overline{\lim}\, t_n = \underline{\lim}\, t_n = 0,$$

and so

$$t_n \to 0.$$

That is to say $s_n \to s$, *and so* $\sum_{n=1}^{\infty} a_n$ *is convergent and its sum is* s.

If a_n be complex, we consider $R(a_n)$ and $I(a_n)$ separately, and find that $\sum_{n=1}^{\infty} R(a_n)$ and $\sum_{n=1}^{\infty} I(a_n)$ converge by the theorem just proved, and so $\sum_{n=1}^{\infty} a_n$ converges.

The reader will see in Chapter IX that this result is of great importance in the modern theory of Fourier series.

Corollary. If $a_n(\xi)$ *be a function of* ξ *such that* $\sum_{n=1}^{\infty} a_n(\xi)$ *is uniformly summable* $(C\,1)$ *throughout a domain of values of* ξ, *and if* $|a_n(\xi)| < Kn^{-1}$, *where* K *is independent of* ξ, $\sum_{n=1}^{\infty} a_n(\xi)$ *converges uniformly throughout the domain.*

For, retaining the notation of the preceding section, if $t_n(\xi)$ does not tend to zero uniformly, we can find a positive number h independent of n and ξ such that an infinite sequence of values of n can be found for which $t_n(\xi_n) > h$ or $t_n(\xi_n) < -h$ for some point ξ_n of the domain*; the value of ξ_n depends on the value of n under consideration.

We then find, as in the original theorem,

$$\tfrac{1}{2}h^2K^{-1}n < n\,.\,o(1)$$

for a set of values of n tending to infinity. The contradiction implied in the inequality shews† that h does not exist, and so $t_n(\xi) \to 0$ uniformly.

* It is assumed that $a_n(\xi)$ is real; the extension to complex variables can be made as in the former theorem. If no such number h existed, $t_n(\xi)$ would tend to zero uniformly.

† It is essential to observe that the constants involved in the inequality do not depend on ξ_n. For if, say, K depended on ξ_n, K^{-1} would really be a function of n and might be $o(1)$ *qua* function of n, and the inequality would not imply a contradiction.

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Miscellaneous Examples.

1. Shew that $\int_0^\infty \frac{e^{-xt}}{1+t^2}dt \sim \frac{1}{x}-\frac{2!}{x^3}+\frac{4!}{x^5}-\ldots$ when x is real and positive.

2. Discuss the representation of the function

$$f(x)=\int_{-\infty}^{0} \phi(t)\, e^{tx}\, dt$$

(where x is supposed real and positive, and ϕ is a function subject to certain general conditions) by means of the series

$$f(x)=\frac{\phi(0)}{x}-\frac{\phi'(0)}{x^2}+\frac{\phi''(0)}{x^3}-\ldots.$$

Shew that in certain cases (e.g. $\phi(t)=e^{at}$) the series is absolutely convergent, and represents $f(x)$ for large positive values of x; but that in certain other cases the series is the asymptotic expansion of $f(x)$.

3. Shew that

$$e^z z^{-a}\int_z^\infty e^{-x}x^{a-1}\,dx \sim \frac{1}{z}+\frac{a-1}{z^2}+\frac{(a-1)(a-2)}{z^3}+\ldots$$

for large positive values of z.

(Legendre, *Exercices de Calc. Int.* (1811), p. 340.)

4. Shew that if, when $x>0$,

$$f(x)=\int_0^\infty \left\{\log u+\log\left(\frac{1}{1-e^{-u}}\right)\right\} e^{-xu}\frac{du}{u},$$

then

$$f(x)\sim\frac{1}{2x}-\frac{B_1}{2^2x^2}+\frac{B_2}{4^2x^4}-\frac{B_3}{6^2x^6}+\ldots.$$

Shew also that $f(x)$ can be expanded into an absolutely convergent series of the form

$$f(x)=\sum_{k=1}^{\infty}\frac{c_k}{(x+1)(x+2)\ldots(x+k)}.$$

(Schlömilch.)

5. Shew that if the series $1+0+0-1+0+1+0+0-1+\ldots$, in which two zeros precede each -1 and one zero precedes each $+1$, be 'summed' by Cesàro's method, its sum is $\frac{3}{5}$.

(Euler, Borel.)

6. Shew that the series $1-2!+4!-\ldots$ cannot be summed by Borel's method, but the series $1+0-2!+0+4!+\ldots$ can be so summed.

* This paper contains many references to recent developments of the subject.

† A bibliography of the literature of summable series will be found on p. 372 of this memoir.