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# 7.4: The Photon

Maxwell's equations in gaussian units are: & $c=1$

i.) $\vec{\nabla}\cdot\vec{E} = 4\pi\rho$

ii.) $\vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0$

iii.) $\vec{\nabla}\cdot\vec{B} = 0$

iv) $\vec{\nabla}\times\vec{B} - \frac{\partial\vec{E}}{\partial t} = 4\pi\vec{J}$

where $\vec{J}$ is the current density

$\rho$ is the charge density

$\vec{E}$ & $\vec{B}$ are the electric & magnetic fields respectively

To write Maxwell's equations in 4-vectors, we must first define notations

$$J^{\mu} = (c\rho, \vec{J})$$

To show that this still gives current conservation:

$$\partial_{\mu} J^{\mu} = 0 \quad (\text{eq } 7.74)$$

$$\frac{\partial J^0}{\partial x^0} + \frac{\partial J^1}{\partial x^1} + \frac{\partial J^2}{\partial x^2} + \frac{\partial J^3}{\partial x^3}$$

$$= \frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$$

Next, we define the field tensor

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}$$

This is different from the book by a minus sign, but is a matter of notation & will yield the same results.

Another property of this matrix is:

$$F^{\nu\mu} = -F^{\mu\nu}$$

so that $F^{\nu\mu}$ is an antisymmetric 2nd rank tensor.

Next, we define the dual strength tensor,

$$G^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z & E_y \\ -B_y & E_z & 0 & -E_x \\ -B_z & -E_y & E_x & 0 \end{pmatrix}$$

which can be obtained from $F^{\mu\nu}$ by substituting

$\vec{E} \to \vec{B}$ & $\vec{B} \to -\vec{E}$

Using this notation, Maxwell's can be written as:

$$\frac{\partial F^{\mu\nu}}{\partial x^\nu} = \frac{4\pi}{c} J^\nu \quad \& \quad \frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0$$

$\hookrightarrow$ eq 7.73

using $\frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0$

for eq iii)

$\mu = 0$

$$\frac{\partial G^{0\nu}}{\partial x^\nu} = \frac{\partial G^{00}}{\partial x^0} + \frac{\partial G^{01}}{\partial x^1} + \frac{\partial G^{02}}{\partial x^2} + \frac{\partial G^{03}}{\partial x^3}$$

$$= \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = \vec{\nabla} \cdot \vec{B} = 0$$

for eq ii)

$\mu = 1$

$$\frac{\partial G^{1\nu}}{\partial x^\nu} = \frac{\partial G^{10}}{\partial x^0} + \frac{\partial G^{11}}{\partial x^1} + \frac{\partial G^{12}}{\partial x^2} + \frac{\partial G^{13}}{\partial x^3}$$

$$= -\frac{\partial \vec{B}_x}{\partial t} - \frac{\partial E_z}{\partial y} + \frac{\partial E_y}{\partial z}$$

$$= \left(+\frac{\partial \vec{B}}{\partial t} + \vec{\nabla} \times \vec{E}\right)_x = 0$$

doing $\mu = 2$ & 3 and combining w/ $\mu = 1$,

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

Next, we define a potential for $\vec{B}$, given as

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

Notice that $\vec{\nabla} \cdot \vec{B} = 0$ w/ this potential

$$\vec{\nabla} \cdot \vec{B} = \underbrace{\vec{\nabla} \cdot \vec{\nabla} \times \bar{A}}_{=0}$$

Plugging this into equation ii.)

$$\vec{\nabla} \times \vec{E} + \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{A}) = 0$$

Explicitly carrying (sp?) out these equations, we see we get Maxwell's equation back

for equation i),

$\mu = 0$

$$\frac{\partial F^{0\nu}}{\partial x^\nu} = \frac{\partial F^{00}}{\partial x^0} + \frac{\partial F^{01}}{\partial x^1} + \frac{\partial F^{02}}{\partial x^2} + \frac{\partial F^{03}}{\partial x^3}$$

$$= \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right)$$

$$= \left(\vec{\nabla}\cdot\vec{E}\right)$$

$$\frac{4\pi}{c} J^0 = \frac{4\pi}{c} c\rho$$

$$= 4\pi\rho$$

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho$$

For equation iv)

$\mu = 1$

$$\frac{\partial F^{1\nu}}{\partial x^\nu} = \frac{\partial F^{10}}{\partial x^0} + \frac{\partial F^{11}}{\partial x^1} + \frac{\partial F^{12}}{\partial x^2} + \frac{\partial F^{13}}{\partial x^3}$$

$$= -\frac{\partial E_x}{\partial t} + \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} + \frac{\partial 0}{\partial x}$$

$$4\pi J^1 = 4\pi J_x$$

$$\left(-\frac{\partial \vec{E}}{\partial t} + \vec{\nabla}\times\vec{B}\right)_x = 4\pi J_x$$

continuing this for $\mu = 2$ & $3$, you get the rest of the components, resulting w/

$$-\frac{\partial \vec{E}}{\partial t} + \vec{\nabla}\times\vec{B} = 4\pi\vec{J}$$

since $\vec{\nabla}$ is independent of time,

$$\vec{\nabla} \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0$$

as the curl vanishes, it can be written as the gradient of a scalar.

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla} V$$

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} \quad (\text{eq } 7.77)$$

In 4-vector notation,

$$A^\mu = (V, A_x, A_y, A_z)$$

& this can be written as

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

$$= \frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu}$$

Showing that this gives the correct eq

$$F^{01} = \frac{\partial A^1}{\partial x_0} - \frac{\partial A^0}{\partial x_1} = -\frac{\partial A_x}{\partial t} - \frac{\partial V}{\partial x}$$

$$= \left(-\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}\right)_x = \vec{E}_x$$

doing the y & z, we get

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}$$

plugging this into our previous equation 7.73

$$\partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu = 4\pi J^\nu$$

7.18)

a) Derive eq 7.70 (i + iv) from 7.73 (Done in ne

b.) Prove eq 7.74 from eq 7.73

$$\partial_\mu F^{\mu\nu} = 4\pi J^\nu$$

$$\partial^\nu g(\partial_\mu F^{\mu\nu}) = \partial_\nu 4\pi J^\nu$$

$$= \partial_\nu \partial_\mu F^{\mu\nu} = 4\pi \partial_\nu J^\nu$$

can be rewritten using antisymmetric properties

$$-\partial_\nu \partial_\mu F^{\nu\mu} = 4\pi \partial_\nu J^\nu$$

only way this can be true is if

$$-\partial_\nu \partial_\mu F^{\nu\mu} = \partial_\nu \partial_\mu F^{\mu\nu} = 0$$

$$\Rightarrow \partial_\nu J^\nu = 0$$

7.19)

eq 7.74 $\partial_\mu J^\mu = 0$

$$\int (\vec{\nabla} \cdot \vec{A})\, d\tau = \oint \vec{A} \cdot d\vec{a}$$

$$\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$$

$$\int (\vec{\nabla} \cdot \vec{J})\, d\tau = -\frac{\partial}{\partial t} \int \rho \, d\tau$$

$$I \text{ flowing out} = \oint \vec{J} \cdot d\vec{a} = -\frac{\partial q}{\partial t}$$

$$-\frac{\partial q}{\partial t} = -\frac{\partial q}{\partial t}$$