Don

The inhomogeneous Maxwell eqns in terms of the 4-vector potential:

$$\partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu = \frac{4\pi}{c} J^\nu$$

However, $A$ is not uniquely defined — under a transformation

$$A^\mu \to A^\mu + \partial^\mu \lambda$$ where $\lambda$ is a scalar function $\lambda = \lambda(\vec{r}, t)$,

the fields are invariant. This is a gauge transformation.

$$\partial_\mu \partial^\mu (A^\nu + \partial^\nu \lambda) - \partial^\nu \partial_\mu (A^\mu + \partial^\mu \lambda) =$$

$$= \partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu + \partial_\mu \partial^\mu \partial^\nu \lambda - \partial^\nu \partial_\mu \partial^\mu \lambda$$

Like $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$, etc, these operators commute.

$$= \partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu$$

$$= \frac{4\pi}{c} J^\nu$$

We can take advantage of this invariance to apply a constraint on $A$.

$$\partial^\mu A_\mu = 0$$ Lorenz condition

Applying this condition we get

$$\partial_\mu \partial^\mu A^\nu = \frac{4\pi}{c} J^\nu = \Box A^\nu$$

Where $\Box \equiv \partial_\mu \partial^\mu = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \vec{\nabla}^2$, is the D'Alembertian operator.

Even with this condition, $A$ is not uniquely specified; it is invariant under a gauge transformation for which $\lambda$ satisfies

$$\Box \lambda = 0$$

---

$$\Box(A^\nu + \partial^\nu \lambda) = \Box A^\nu + \Box \partial^\nu \lambda$$

$$= \Box A^\nu + \partial_\mu \partial^\mu \partial^\nu \lambda$$

$$= \Box A^\nu + \partial^\nu \underbrace{\partial_\mu \partial^\mu \lambda}$$

$$= \Box A^\nu = \frac{4\pi}{c} J^\nu \qquad \Box \lambda = 0$$

---

We can either keep this ambiguity, or we can apply another condition — one that will break the Lorentz-invariance of $A$. We will do the latter.

In empty space, we let

$$A^0 = 0 \quad \text{so that} \quad \vec{\nabla} \cdot \vec{A} = 0.$$

This is the Coulomb gauge condition. It is <u>not</u> Lorentz-invariant, so any Lorentz transformation will require a corresponding gauge transformation to restore the Coulomb gauge.

Up to this point, we've been looking at classical electrodynamics,

In quantum electrodynamics, the 4-vector potential is significant — specifically, it is the wavefunction of the photon.

In free space, ($J^\mu = 0$),

$$\Box A^\mu = 0, \text{ or } \frac{1}{c^2}\frac{\partial^2}{\partial t^2}A^\mu - \vec{\nabla}^2 A^\mu = 0$$

This is the Klein-Gordon equation for a massless particle (i.e. the photon)

We'll look for plane-wave solutions of the form

$$A^\mu(x) = a e^{-i/\hbar \, p^\nu x_\nu} \epsilon^\mu(p),$$

where $\epsilon^\mu$ is the polarization vector, which characterizes the spin of the photon.

Applying the Klein-Gordon eqn to this potential:

$$\Box A^\mu = \partial_\lambda \partial^\lambda \left(a e^{-i/\hbar \, p^\nu x_\nu} \epsilon^\mu\right)$$

$$0 = a\epsilon^\mu \partial_\lambda e^{-i/\hbar \, p^\nu x_\nu}\left(-\frac{i}{\hbar} p^\nu g_{\lambda\nu}\right)$$

$$0 = a\epsilon^\mu e^{-i/\hbar \, p^\nu x_\nu}\left(-\frac{1}{\hbar^2}\right) p^\nu p_\nu$$

$$0 = p^\nu p_\nu = m_\gamma^2 = \frac{E^2}{c^4} - \frac{|\vec{p}|^2}{c^2}$$

$$\therefore E = |\vec{p}|c$$

Applying the Lorenz condition,

$$\partial_\mu A^\mu = 0$$

$$= \partial_\mu a e^{-i/\hbar p^\nu x_\nu} \epsilon^\mu(p)$$

$$0 = a e^{-i/\hbar p^\nu x_\nu} \left(-\frac{i}{\hbar} p^\nu g_{\mu\nu}\right) \epsilon^\mu(p)$$

$$0 = p_\mu \epsilon^\mu$$

---

This sets a condition on $\epsilon^\mu(p)$.

Furthermore, in the Coulomb gauge,

$A^0 = 0$ and therefore $\epsilon^0 = 0$, and $\vec{\epsilon} \cdot \vec{p} = 0$ (3-vectors)

With these two conditions, we are left with <u>two</u> independent solutions for $\epsilon$, both perpendicular to the 3-momentum.

The photon is massless and only has two spin states, ($m_s = \pm 1$) not three.

Note that if we do not apply the Coulomb gauge condition, we get three solutions. One of the solutions, corresponding to $m_s = 0$, is unphysical.

Example: photon moving in the z-direction.

If we do not apply the Coulomb gauge condition,

$$p = (|\vec{p}|, 0, 0, |\vec{p}|), \quad 0 = p_\mu \epsilon^\mu$$

$\uparrow$ $E = |\vec{p}|c$

Three solutions exist:

$$\epsilon^{(1)} = (0, 1, 0, 0)$$
$$\epsilon^{(2)} = (0, 0, 1, 0)$$
$$\epsilon^{(3)} = \frac{1}{\sqrt{2}}(-1, 0, 0, 1)$$

longitudinal free photon

There is an extra, unphysical solution corresponding to $m_s = 0$.

If we apply the Coulomb gauge, $\epsilon^0 = 0$, leaving two solutions,

$$\epsilon^{(1)} = (0, 1, 0, 0)$$
$$\epsilon^{(2)} = (0, 0, 1, 0)$$

---

Note that the spin eigenstates of the photon are circularly polarized, with

$$\epsilon_\pm = \mp(\epsilon_1 \pm i\epsilon_2)/\sqrt{2}, \quad m_s = \pm 1.$$

In the above example,

$$\epsilon_+ = \frac{1}{\sqrt{2}}(0, -1, -i, 0) \text{ and } \epsilon_- = \frac{1}{\sqrt{2}}(0, 1, -i, 0)$$