Chapter 7: QED

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7.1 First Half → Klein-Gordon Equation

On Monday, Tao introduced the Toy Theory, which told us of an idealized quantum field theory with particles of spin zero. What about $s = \frac{1}{2}, 1$, etc...

→ Section 7.1 describes the EOM for particles of spin $0, \frac{1}{2}$.

I will discuss the EOM for a spin 0 particle and Nima will pick up from there in one week by giving the Dirac Equation, that which governs spin $\frac{1}{2}$.

Schrödinger gave the non-relativistic EOM

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = i\hbar\frac{\partial\psi}{\partial t} \qquad P \to \frac{\hbar}{i}\vec{\nabla}$$

$$H\psi = E\psi \qquad E \to i\hbar\frac{\partial}{\partial t}$$

Deduced from the correspondence priciple from the Hamiltonian formalism of N.R. C.M. $(v \ll c)$

→ Invariant under Galilean transformations.

Relativistic QM

We desire an equation that remains Lorentz invariant.

We want a complete theory that must encompass, in a single scheme, the dynamical states differing not only by the quantum state, but also by the # and nature of the elementary particles.

Some Early Problems:

The law of conservation of #'s of particles ceases in general to be true.

$$E = mc^2$$

Leads to:
Creation & Absorption of particles whenever the interactions gives rise to energy transfers equal or superior to the rest mass of these particles.

What follows is a slight deviation from griffith but the end result is the same.

π^--meson, $s=0$

Since a free particle is boring, let's choose a particle with zero spin, charge e in a vector potential $\vec{A}(\vec{r},t)$ and a scalar potential $\phi(\vec{r},t)$ such that:

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

N.B. $\hbar = c = 1$

Let's write down the classical-relativistic Hamiltonian, then make the correspondence

$$E \rightarrow i\frac{\partial}{\partial t} \quad , \quad p \rightarrow -i\vec{\nabla}$$

$$H = P_i V_i - L$$

Now Find L: We can use the Lagrangian equation of motion and work backwards to get the Lagrangian.

We have particle with constant rest mass

$m_0 = \text{const.}$

that is in motion relative to an inertial frame under the action of a force derivable from a potential V. $v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$

$$\Rightarrow \vec{F} = -\frac{\partial V}{\partial x} = \frac{d\vec{P}}{d\tau} = \frac{d}{d\tau}(m_0 \dot{x})$$

$$= \frac{d}{dt}\left[\frac{m_0 \dot{x}}{(1-v^2)^{1/2}}\right]$$

$$\Rightarrow \frac{d}{dt}\left[\frac{m_0 \dot{x}}{(1-v^2)^{1/2}}\right] = -\frac{\partial V}{\partial x} = \frac{\partial L}{\partial x}$$

EOM: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$ or $\frac{\partial L}{\partial x} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)$

$$= \frac{d}{dt}\left[\frac{m_0 \dot{x}}{(1-v^2)^{1/2}}\right]$$

$$\therefore \underline{L = -m_0(1-v^2)^{1/2} - V} \quad \text{satisfies EOM}$$

$$H = \dot{p}_i v_i - L = p_x v_x + p_y v_y + p_z v_z - L$$

$$= p_x v_x + p_y v_y + p_z v_z + m_0 (1 - v^2)^{1/2} + V(\vec{r}, t)$$

where $p_i = \gamma m_0 v_i$

$$= \gamma m_0 v_x^2 + \gamma m_0 v_y^2 + \gamma m_0 v_z^2 + m_0 (1 - v^2)^{1/2} + V(\vec{r}$$

$$= \gamma m_0 \underbrace{(v_x^2 + v_y^2 + v_z^2)}_{v^2} + m_0 (1 - v^2)^{1/2} + V(\vec{r}, t)$$

$$= \frac{m_0 v^2}{(1 - v^2)^{1/2}} + m (1 - v^2)^{1/2} + V(\vec{r}, t)$$

$$= \frac{m_0 \cancel{v^2}}{(1 - \cancel{v^2})^{1/2}} + \frac{m_0 (1 - \cancel{v^2})}{(1 - v^2)^{1/2}} + V(\vec{r}, t)$$

$E = \sqrt{\tilde{p}^2 + m_0^2}$

$$\therefore H = \frac{m_0}{(1 - v^2)^{1/2}} + V(\vec{r}, t)$$

$$= \underline{\sqrt{\tilde{p}^2 + m_0^2} + V(\vec{r}, t)}$$

where $\tilde{p} = \vec{p} - e\vec{A}$ and $V(\vec{r}, t) = e\phi(\vec{r}, t)$

$$\boxed{\therefore H = \sqrt{(\vec{p} - e\vec{A})^2 + m_0^2} + e\phi(\vec{r}, t)}$$

Now Find the Hamiltonian for a particle in an electromagnetic field.

$$L = -m_0 (1-v^2)^{1/2} - V$$

where $V = e\phi - e\vec{A}\cdot\vec{v}$

$$L = -m_0(1-v^2)^{1/2} + e\vec{A}\cdot\vec{v} - e\phi$$

$$H = \vec{P}\cdot\vec{v} - L$$ where $\tilde{p} = \gamma m_0 \vec{v}$

$$\vec{P} = \frac{m_0\vec{v}}{(1-v^2)^{1/2}} + e\vec{A}$$

$$= \gamma m_0 \vec{v} + e\vec{A} \equiv$$

Now correspond $E \to i\frac{\partial}{\partial t}$, $p \to -i\vec{\nabla}$

$$E = \sqrt{(\vec{p} - e\vec{A})^2 + m_0^2} + e\phi$$

$$\underbrace{E - e\phi}_{\tilde{E}} = \sqrt{(\vec{p} - e\vec{A})^2 + m_0^2}$$

Allow them to operate on a wavefunction, ψ

$$\Rightarrow \left(i\frac{\partial}{\partial t} - e\phi\right)\psi = \left[(-i\vec{\nabla} - e\vec{A})^2 + m_0^2\right]^{1/2}\psi$$

Right away we can notice the dissymmetry both the space and time coordinates. We would like to find a relativistic invariance. Plus, the right-hand operater is very difficult as a square root, except when the field vanishes.

$$\Rightarrow m_0^2 = \tilde{E}^2 - \tilde{p}^2 = (E - e\phi)^2 - (\vec{p} - e\vec{A})^2$$

$$\Rightarrow \left[\left(i\frac{\partial}{\partial t} - e\phi\right)^2 - (-i\vec{\nabla} - e\vec{A})^2\right]\psi = m_0^2\psi$$

N.B.

$$m_0^2 = (E - e\phi)^2 - (\vec{p} - e\vec{A})^2$$

$$(E - e\phi)^2 = (\vec{p} - e\vec{A})^2 + m_0^2$$

$$E = e\phi \pm \sqrt{(\vec{p} - e\vec{A})^2 + m_0^2}$$

Where we take the "+" sign for positive mass. That only corresponds to real solutions. The "−" sign corresponds to negative mass and has no physical significance.

Define $A^\mu = (\phi, \vec{A})$ and $\partial^\mu = \left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \right.$

$$= \left(\frac{\partial}{\partial t}, \vec{\nabla}\right)$$

$$\underline{D^\mu = (\partial^\mu + ieA^\mu)}$$

$$\Rightarrow D_\mu D^\mu = (\partial_\mu + ieA_\mu)(\partial^\mu + ieA^\mu)$$

$$= \left(\frac{\partial}{\partial t} + ie\phi\right)^2 - (\vec{\nabla} - ie\vec{A})^2$$

$$= -\left[\left(i\frac{\partial}{\partial t} - e\phi\right)^2 - (-i\vec{\nabla} - e\vec{A})^2\right]$$

$$\boxed{\therefore \left[D_\mu D^\mu + m_0^2\right]\Psi = 0}$$

If we let the fields vanish, then

$$\boxed{\left[\Box + m_0^2\right]\Psi = 0 \qquad \text{(Griffiths)}}$$

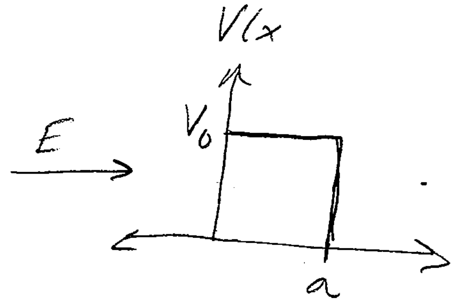
where $\Box \equiv$ D'Alembertian Operator

$$\equiv \partial_\mu \partial^\mu = \frac{\partial^2}{\partial t^2} - \vec{\nabla}^2$$

Klein Paradox

Scatter a relativistic particle off of a potential barrier such that:

$$V(x) \begin{cases} V_0 \ , & |x| < a \\ 0 \ , & |x| > a \end{cases}$$



1-D for simplicity.

$\Rightarrow$ At the barrier we have the EOM

$$\left[\left(i\frac{\partial}{\partial t} - V(x)\right)^2 - \frac{\partial^2}{\partial x^2} + m_0^2\right]\psi = 0$$

Let's get this of the form:

$$\left[\frac{\partial^2}{\partial x^2} + 2m\left(E_{eff} - V_{eff}\right)\right]\psi = 0 \qquad \text{(Schrödinger's Eq)}$$

$$\Rightarrow \left[\frac{\partial^2}{\partial x^2} - \left(i\frac{\partial}{\partial t} - V_0\right)^2 - m_0^2\right]\psi = 0$$

Recall $E \to i\frac{\partial}{\partial t}$

$$\left[\frac{\partial^2}{\partial x^2} - (E - V_0)^2 - m_0^2\right]\psi = 0$$

$$\left\{\frac{\partial^2}{\partial x^2} - 2m_0\left[\frac{(E-V_0)^2 - m_0^2}{m_0}\right]\right\}\psi = 0$$

$$\Rightarrow \left\{ \frac{\partial^2}{\partial x^2} - 2m_0 \left[\frac{E^2 - m_0^2}{2m_0} - \frac{2EV_0 - V_0^2}{2m_0} \right] \right\} \psi = 0$$

$$\Rightarrow E_{eff} = \frac{E^2 - m_0^2}{2m_0} \qquad V_{eff} = \frac{2EV_0 - V_0^2}{2m_0}$$

Conside a strong potential, $\underline{V_0 > 2E}$

$\Rightarrow V_{eff} < 0 \rightarrow$ Bound State

The barrier becomes a well!

Rather than being scattered, the particle is trapped.

References:

* Introduction to Elementary Particles — Griffiths
* Quantum Mechanics — Messiah
* Intro to Tensor Calc, Relativity, and Cosmology — Lawden
* Classical Mechanics — Goldstein
* Rao, N.A., and Kagali, B.A., arXiv: quant-ph/0607223v1, Jul 2006
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