Chapter 8

Constraints. Noninertial coordinate systems

8.1 Constraints

Oftentimes we encounter problems with constraints. For example, for a ball rolling on a floor without slipping, there is a constraint linking the velocity of the center of mass of the ball with the angular velocity of the ball about its center of mass. Often these constraints can be built into the choice of generalized coordinates, but not always, and there are other cases where more insight is obtained by imposing the constraint explicitly.

A *holonomic* constraint is one in which there is a condition between the generalized coordinates, say

$$\Phi(\{q_a\}) = 0. \tag{8.1}$$

Suppose the constraint involves the velocities linearly,

$$\sum_a F_a(\{q_a\})\dot{q}_a = 0, \tag{8.2}$$

or

$$\sum_a F_a(\{q_a\})dq_a = 0. \tag{8.3}$$

This is of the form of the holonomic constraint if $F_a = \frac{\partial \Phi}{\partial q_a}$ and if the constraint doesn't explicitly depend on time; if this is not true, this is called a *nonholonomic constraint.* For example, for a cylinder or a sphere rolling on a plane without slipping, there is a constraint between the velocity of the center of mass $\mathbf{V}$ and the angular velocity of rotation $\mathbf{\Omega}$, because the instantaeous point of contact is always at rest,

$$\mathbf{V} - a\mathbf{\Omega} \times \mathbf{n} = 0, \tag{8.4}$$

where a is the radius of the cylinder or sphere, and $\mathbf{n}$ is a unit vector from the axis of the cylinder or the center of the sphere and the point of contact. For a cylinder this is a holonomic constraint, because $\mathbf{\Omega} = \hat{\mathbf{z}}\dot{\phi}$, where ϕ is the angle of rotation about the cylinder axis, and $\hat{\mathbf{z}}$ is the fixed direction of the axis of the cylinder. The constaint can then be integrated:

$$\frac{d}{dt}(X - a\phi) = 0, \quad \text{or} \quad X - a\phi = \text{constant}. \tag{8.5}$$

Here X is the position of the center of the cylinder, which is free to move on the plane in a direction perpendicular to its axis. The same constraint for a sphere on a plane is nonholonomic, because the sphere, having but one point of contract, is free to move in two dimensions, so the constraint cannot be integrated.

One standard way of dealing with constraints is the method of Lagrange multipliers. Because of the constraints, the coordinates in the action principle are not independent, so we cannot perform independent variations. So what we do is add to the variation of the action an additional term

$$\delta W + \int_2^1 dt \sum_\alpha \lambda_\alpha(\{q_a\}) \sum_a F_{\alpha a}(\{q_a\})\delta q_a = 0. \tag{8.6}$$

Here we have assumed there may be several constraints, labelled by the index α. The λ_α's are arbitrary functions of the coordinates. The constaints would imply that the extra terms added to the action are zero, so we retain the original statement of invariance if there are no endpoint variations. But now the idea is to relax that constaint, but instead regard all the q_a's as independent, and then let the resulting equations determine the functions λ_α, the Lagrange multipliers. This then gives the following modified Lagrange equations,

$$\dot{p}_a = \frac{\partial L}{\partial q_a} + \sum_\alpha \lambda_\alpha F_{\alpha a}. \tag{8.7}$$

There is exactly one Lagrange multiplier for each constraint, so there are exactly the right number of equations to determine the motion and the multipliers.

Let us illustrate this with an elementary example, a sphere of mass m and radius a rolling down an inclined plane without slipping. Here we are considering motion in a single direction, so the nonholonomic remark above is irrelevant. Let the plane angle to the horizontal be α and let the coordinate of the center of mass of the sphere parallel to the inclined plane be x. If the angular velocity of the sphere is $\omega = \dot{\phi}$, the Lagrangian is

$$L = \frac{1}{2}m\dot{x}^2 + mgx\sin\alpha + \frac{1}{2}I\omega^2, \tag{8.8}$$

where for a sphere, we recall, $I = \frac{2}{5}ma^2$. Using the notation above for the constraint $\dot{x} - a\dot{\phi}$, we have $F_1 = 1$, $F_2 = -a$, so the modified Lagrange equations are

$$m\ddot{x} = mg\sin\alpha + \lambda, \quad \frac{2}{5}ma^2\ddot{\phi} = -\lambda a. \tag{8.9}$$

Multiplying the first equation by a, adding, and using the constraint, we get the familiar result,

$$\frac{7}{5}\ddot{x} = g\sin\alpha. \tag{8.10}$$

We note that λ here actually has a direct physical meaning: $-\lambda$ is the frictional force on the sphere at the point of contact that heeps the sphere from slipping, and provides the torque about its center of mass.

8.1.1 Spherical Pendulum

Consider a spherical pendulum, a mass point suspending on a string of length l, so the particle is constrained to move on the surface of a sphere. In Cartesian coordinates, centered at the point of suspension, the Lagrangian is

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz. \tag{8.11}$$

The constraint is

$$F = \frac{1}{2}(x^2 + y^2 + z^2 - l^2) = 0, \quad \text{or} \quad dF = xdx + ydy + zdz = 0. \tag{8.12}$$

Thus, our Lagrangian equations of motion with the contraint encoded with the help of a Lagrange multiplier λ, is

$$\frac{d}{dt}p_a = \frac{\partial L}{\partial q_a} + \lambda\frac{\partial F}{\partial q_a}, \tag{8.13}$$

or explicitly,

$$m\ddot{x} = \lambda x, \quad m\ddot{y} = \lambda y, \quad m\ddot{z} = -mg + \lambda z. \tag{8.14}$$

Combining the first two equations gives

$$ym\ddot{x} - xm\ddot{y} = \frac{d}{dt}m(y\dot{x} - x\dot{y}) = 0, \tag{8.15}$$

which is to say that the angular momentum about the z axis is conserved,

$$\dot{L}_z = 0, \quad L_z = xp_y - yp_x. \tag{8.16}$$

And, multiplying each equation of motion by the corresponding velocity component, gives the energy:

$$m(\dot{x}\ddot{x} + \dot{y}\ddot{y} + \dot{z}\ddot{z}) = -mg\dot{z} + \lambda(x\dot{x} + y\dot{y} + z\dot{z}), \tag{8.17}$$

or

$$\dot{E} = 0, \quad E = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + mgz, \tag{8.18}$$

which uses the constraint equation (8.12). If, instead, we multiply each equation of motion by the corresponding coordinate, we obtain

$$m(x\ddot{x} + y\ddot{y} + z\ddot{z}) = \lambda(x^2 + y^2 + z^2) - mgz, \tag{8.19}$$

or

$$\lambda l = m\left(\ddot{x}\frac{x}{l} + \ddot{y}\frac{y}{l} + \ddot{z}\frac{z}{l}\right) + mg\frac{z}{l}. \tag{8.20}$$

But the (outward) normal to the spherical surface is

$$\hat{\mathbf{n}} = \left(\frac{x}{l}, \frac{y}{l}, \frac{z}{l}\right), \tag{8.21}$$

so

$$\lambda l = \mathbf{F}\cdot\hat{\mathbf{n}} - \mathbf{F}_g\cdot\hat{\mathbf{n}}, \tag{8.22}$$

where $\mathbf{F}$ is the total force on the mass point, and $\mathbf{F}_g$ is the gravitational force. Thus, the significance of the Lagrange multiplier is that λl is the tension in the string.

For completeness, let us proceed to solve this problem, although it was treated in one of the homework problems. In spherical polar coordinates,

$$x = l\sin\theta\cos\phi, \quad y = l\sin\theta\sin\phi, \quad z = l\cos\theta, \tag{8.23}$$

the constraint is automatically satisfied, and the energy is

$$E = \frac{1}{2}ml^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) + mgl\cos\theta, \tag{8.24}$$

and the conserved z-component of angular momentum is

$$L_z = p_\phi = ml^2\sin^2\theta\dot{\phi}. \tag{8.25}$$

Using $u = \cos\theta$, we can rewrite the energy equation as

$$\dot{u}^2 = \frac{2}{ml^2}[E - mglu](1-u^2) - \frac{L_z^2}{m^2l^4} \equiv U(u), \tag{8.26}$$

which is integrated to read

$$t = \int \frac{du}{\sqrt{U}}. \tag{8.27}$$

The motion is confined between two turning points, $-1 < u_1 < u < u_2 < 1$, and the period of motion is

$$T = 4\int_{u_1}^{u_2} \frac{du}{U(u)}, \tag{8.28}$$

which is an elliptic integral of the first kind.

At the same time the pendulum bob is oscillating back and forth in θ it is precessing in ϕ. According to Eq. (8.25),

$$\dot{\phi} = \frac{L_z}{ml^2(1-u^2)}, \tag{8.29}$$

and so

$$d\phi = \frac{L_z}{ml^2(1-u^2)}\frac{du}{\sqrt{U}}. \tag{8.30}$$

Therefore, the angle of precession in one period is

$$2\pi + \Delta\phi = \frac{4L_z}{ml^2}\int_{u_1}^{u_2}\frac{du}{(1-u^2)\sqrt{U}}. \tag{8.31}$$

8.2 Noninertial coordinate systems

In discussing rigid bodies, we went from an inertial coordinate system, to another system based at the center of mass of the body, and then to a rotated coordinate system. Let's abstract from that. Start from an inertial coordinate system K_0, where the Lagrangian of a single particle has the form

$$L_0 = \frac{1}{2}mv_0^2 - U(\mathbf{r}). \tag{8.32}$$

First, go to another coordinate system K' with axes parallel to those of K_0, but which is moving relative to the first with a prescribed velocity $\mathbf{V}(t)$. Then if $\mathbf{v}'$ represents the velocity of the particle in K',

$$\mathbf{v}_0 = \mathbf{v}' + \mathbf{V}(t), \quad \mathbf{r}_0 = \mathbf{r}' + \mathbf{R}(t), \tag{8.33}$$

so the Lagrangian becomes

$$L_0 = \frac{1}{2}mv'^2 + m\mathbf{v}' \cdot \mathbf{V} + \frac{1}{2}mV^2 - U. \tag{8.34}$$

Now the third term, as a prescribed function of t, can be written as a total time derivative, so is irrelevant to the equations of motion:

$$\mathbf{V}(t)^2 = \frac{d}{dt}F(t), \quad F(t) = \int^t dt' V(t')^2, \tag{8.35}$$

so we can replace the Lagrangian by

$$L' = \frac{1}{2}mv'^2 + m\mathbf{v}' \cdot \mathbf{V} - U. \tag{8.36}$$

The canonical momentum is unchanged:

$$\mathbf{p}' = \frac{\partial L}{\partial \mathbf{v}'} = m\mathbf{v}' + m\mathbf{V} = m\mathbf{v}_0 = \mathbf{p}_0, \tag{8.37}$$

as is the equation of motion,

$$\dot{\mathbf{p}}' = \dot{\mathbf{p}}_0 = -\frac{\partial U}{\partial \mathbf{r}} = -\frac{\partial U}{\partial \mathbf{r}'}. \tag{8.38}$$

By "integrating by parts" we can remove another total time derivative:

$$m\mathbf{v}' \cdot \mathbf{V} = m\frac{d}{dt}(\mathbf{r}' \cdot \mathbf{V}) - m\mathbf{r}' \cdot \mathbf{A}, \quad \mathbf{A} = \frac{d\mathbf{V}}{dt}, \tag{8.39}$$

in terms of the prescribed acceleration $\mathbf{A}$, so the Lagrangian in the moving frame K' has the form

$$L' = \frac{1}{2}mv'^2 - m\mathbf{r}' \cdot \mathbf{A} - U, \tag{8.40}$$

which leads to the equation of motion

$$\frac{d}{dt}m\mathbf{v}' = -\frac{\partial}{\partial \mathbf{r}}U - m\mathbf{A}(t), \tag{8.41}$$

which is of course true, because

$$\frac{d}{dt}m\mathbf{v}' = \frac{d}{dt}m\mathbf{v} - m\mathbf{A}. \tag{8.42}$$

Now, we move on to the rotating frame K. Now although the position vector is unchanged, $\mathbf{r}' = \mathbf{r}$, the velocity transforms as

$$\mathbf{v}' = \mathbf{v} + \boldsymbol{\Omega} \times \mathbf{r}, \tag{8.43}$$

so the Lagrangian (8.40) becomes

$$L = \frac{1}{2}mv^2 + \frac{1}{2}m(\boldsymbol{\Omega} \times \mathbf{r})^2 + m\mathbf{v} \cdot (\boldsymbol{\Omega} \times \mathbf{r}) - U - m\mathbf{r} \cdot \mathbf{A}. \tag{8.44}$$

Now the momentum is

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}} = m\mathbf{v} + m\boldsymbol{\Omega} \times \mathbf{r}, \tag{8.45}$$

and the equation of motion reads

$$\dot{\mathbf{p}} = m\dot{\mathbf{v}} + m\dot{\boldsymbol{\Omega}} \times \mathbf{r} + m\boldsymbol{\Omega} \times \dot{\mathbf{r}} = \frac{\partial L}{\partial \mathbf{r}} = -\frac{\partial U}{\partial \mathbf{r}} - m\mathbf{A} - m\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) + m\mathbf{v} \times \boldsymbol{\Omega}, \tag{8.46}$$

or

$$m\dot{\mathbf{v}} = -m\dot{\boldsymbol{\Omega}} \times \mathbf{r} + 2m\mathbf{v} \times \boldsymbol{\Omega} + m\boldsymbol{\Omega} \times (\mathbf{r} \times \boldsymbol{\Omega}) - \frac{\partial U}{\partial \mathbf{r}} - m\mathbf{A}. \tag{8.47}$$

The first term on the right is due to any angular acceleration of the frame, the second term is the “Coriolis force,”[1] and the third is the “centrifugal force.” Of course, these are not real forces, but appear because we are expressing the acceleration of a particle in a noninertial coordinate frame.

Let’s finally consider a uniformly rotating coordinate system the origin of which is not accelerated: $\dot{\boldsymbol{\Omega}} = 0$, $\mathbf{A} = 0$. We then compute the energy by making a Legendre transformation,

$$E = \mathbf{p} \cdot \dot{\mathbf{r}} - L = \frac{1}{2}mv^2 - \frac{1}{2}m(\boldsymbol{\Omega} \times \mathbf{r})^2 + U. \tag{8.48}$$

Note that the linear term in $\mathbf{v}$ has dropped out. The second term in the energy is the centrifugal potential energy. Note that the linear momentum and the angular momentum are unchanged by passing to the rotating frame:

$$\mathbf{p}_0 = m\mathbf{v}_0 = m(\mathbf{v} + \boldsymbol{\Omega} \times \mathbf{r}) = \mathbf{p}, \tag{8.49a}$$

$$\mathbf{L}_0 = \mathbf{r} \times \mathbf{p}_0 = m\mathbf{r} \times (\mathbf{v} + \boldsymbol{\Omega} \times \mathbf{r}) = m\mathbf{r} \times \mathbf{p} = \mathbf{L}, \tag{8.49b}$$

[1] Discovered, of course, much earlier, by Riccioli, Grimaldi (1651), and Laplace (1778). Coriolis didn’t publish his paper until 1835.

but the energy is not the same:

$$E = \frac{1}{2}m(-\mathbf{v}_0 + \mathbf{\Omega}\times\mathbf{r})^2 - \frac{1}{2}m(\mathbf{\Omega}\times\mathbf{r})^2 + U = \frac{1}{2}mv_0^2 - m\mathbf{v}_0\cdot\mathbf{\Omega}\times\mathbf{r} + U, \quad (8.50)$$

or

$$E = E_0 - \mathbf{L}\cdot\mathbf{\Omega}. \quad (8.51)$$

Although this result was derived for a single particle, it is quite general, as may be inferred from how the generator of a system of particles changes under a rotation,

$$G = \sum_a \mathbf{p}_a\delta\mathbf{r}_a = \sum_a \mathbf{p}_a\cdot\delta\boldsymbol{\omega}\times\mathbf{r}_a = \delta\boldsymbol{\omega}\cdot\mathbf{L}, \quad \mathbf{L} = \sum_a \mathbf{r}_a\times\mathbf{p}_a. \quad (8.52)$$

Thus for a rotation with angular velocity $\mathbf{\Omega}$ for a time δt, $\delta\boldsymbol{\omega} = \mathbf{\Omega}\delta t$, and the change in the action is the difference of the generators,

$$\delta W = G_1 - G_2, \quad G = -\delta t(E - \mathbf{L}\cdot\mathbf{\Omega}), \quad (8.53)$$

which exhibits exactly the change in the energy seen in Eq. (8.51).

The Hamiltonian is obtained from the energy (8.48) by writing it in terms of the position and the canonical momentum,

$$H(\mathbf{r},\mathbf{p}) = \frac{1}{2m}(\mathbf{p} - m\mathbf{\Omega}\times\mathbf{r})^2 - \frac{1}{2}m(\mathbf{\Omega}\times\mathbf{r})^2 + U = \frac{p^2}{2m} - \mathbf{\Omega}\cdot\mathbf{r}\times\mathbf{p} + U. \quad (8.54)$$

Indeed, it is easy to check that

$$\dot{\mathbf{r}} = \frac{\partial H}{\partial\mathbf{p}} = \frac{\mathbf{p}}{m} - \mathbf{\Omega}\times\mathbf{r} = \mathbf{v}, \quad (8.55)$$

and

$$\dot{\mathbf{p}} = m(\dot{\mathbf{v}} + \dot{\mathbf{\Omega}}\times\mathbf{r} + \mathbf{\Omega}\times\mathbf{v}) = -\frac{\partial H}{\partial\mathbf{r}} = -\mathbf{\Omega}\times\mathbf{p} - \frac{\partial U}{\partial\mathbf{r}}, \quad (8.56)$$

or

$$m\dot{\mathbf{v}} = -m\dot{\mathbf{\Omega}}\times\mathbf{r} - 2m\mathbf{\Omega}\times\mathbf{v} - m\mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r}) - \frac{\partial U}{\partial\mathbf{r}}, \quad (8.57)$$

which agrees with Eq. (8.47).

8.3 Problems for Chapter 8

1. In general, the nodal points of the trajectory of a spherical pendulum advance during the course of the motion. For sufficiently small oscillations, however, the nodal points must be fixed, for we are then dealing with an harmonic elliptical motion. First imagine the motion is confined to a plane, i.e., $L_z = 0$. Compute the period for a small oscillation about the bottom of the sphere. Then let $L_z \neq 0$, but imagine the motion is between two latitudes close to the south pole, $\theta_- \leq \theta \leq \theta_+$, with both angles close to π. Then there are oscillations in θ with period T, and precession in ϕ.

Alternatively, the motion can be described as an elliptical orbit, due to a harmonic restoring force. Compute T and the precession angle $\Delta\phi$ during one complete orbit, in this limit when the area of the elliptical orbit is very small.

[Adapted from Sommerfeld's book.]