Chapter 7

Rigid Body Motion

In this chapter we will discuss the motion of a rigid body. Such an object has a fixed shape and mass distribution. Therefore, it takes six coordinates to specify the system: the three components of the center of mass position vector $\mathbf{R} = (X, Y, Z)$, and the orientation of the body about a frame with the origin at the center of mass of the body, which takes three angles.

First we remind the reader of the definition of the center of mass and relative coordinates. For an arbitrary collection of point masses, labelled by a, with mass m_a and position, relative to some arbitrary origin, $\mathbf{r}_a$, the center of mass position vector is defined by

$$M\mathbf{R} = \sum_a m_a \mathbf{r}_a, \quad M = \sum_a m_a. \tag{7.1}$$

See Fig. 7.1. Let us denote the position of a particle relative to the center of mass by $\mathbf{r}'_a$:

$$\mathbf{r}_a = \mathbf{R} + \mathbf{r}'_a, \tag{7.2}$$

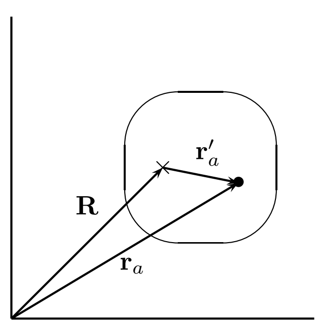

Figure 7.1: Illustration of the center of mass of a distribution of particles, marked with a $\times$, located at position $\mathbf{R}$, and the position vector $\mathbf{r}_a$ of a mass point in the distribution, as well as the position vector $\mathbf{r}'_a$ relative to the center of mass.

so from the definition of the center of mass,

$$\sum_a m_a \mathbf{r}'_a = 0. \tag{7.3}$$

Because this equation holds for all time, the total momentum of the system is

$$\mathbf{P} = \sum_a m_a(\dot{\mathbf{R}} + \dot{\mathbf{r}}'_a) = \sum_a m_a \dot{\mathbf{R}} = M\dot{\mathbf{R}} = M\mathbf{V}, \tag{7.4}$$

where $\mathbf{V}$ is the velocity of the center of mass. Note that the relative momentum sums to zero:

$$\sum_a m_a \dot{\mathbf{r}}'_a = \sum_a \mathbf{p}'_a = 0. \tag{7.5}$$

Now the force acting on a single particle is the sum of forces external to the body and forces due to other particles in the body:

$$\mathbf{F}_a = \mathbf{F}_{a,\mathrm{ext}} + \mathbf{F}_{a,\mathrm{int}}. \tag{7.6}$$

By momentum conservation, the internal forces must cancel out when summed over all the particles, so the total force on the system is just the sum of the external forces:

$$\dot{\mathbf{P}} = M\dot{\mathbf{V}} = \sum_a (\mathbf{F}_{a,\mathrm{ext}} + \mathbf{F}_{a,\mathrm{int}}) = \sum_a \mathbf{F}_{a,\mathrm{ext}} = \mathbf{F}_{\mathrm{ext}}. \tag{7.7}$$

Internal forces cannot cause a body to move. (For two-body forces, this follows from Newton's third law of motion.)

Now consider angular momentum,

$$\begin{aligned}\mathbf{L} &= \sum_a \mathbf{r}_a \times \mathbf{p}_a = \sum_a m_a \mathbf{r}_a \times \mathbf{v}_a = \sum_a m_a (\mathbf{R} + \mathbf{r}'_a) \times (\mathbf{V} + \mathbf{v}'_a) \\ &= M\mathbf{R} \times \mathbf{V} + \sum_a m_a \mathbf{r}'_a \times \mathbf{v}'_a = \mathbf{L}_{\mathrm{CM}} + \mathbf{L}_{\mathrm{rel}}. \end{aligned} \tag{7.8}$$

In the last step we used Eq. (7.3) and its time derivative, Eq. (7.5). The rate of change of angular momentum is

$$\begin{aligned}\dot{\mathbf{L}} &= M\mathbf{R} \times \dot{\mathbf{V}} + \sum_a m_a \mathbf{r}'_a \times \dot{\mathbf{v}}'_a \\ &= \mathbf{R} \times \mathbf{F}_{\mathrm{ext}} + \sum_a \mathbf{r}'_a \times (\mathbf{F}_{a,\mathrm{ext}} + \mathbf{F}_{a.\mathrm{int}}). \end{aligned} \tag{7.9}$$

Again, the internal forces cannot give a net torque on the system, so

$$\dot{\mathbf{L}} = \mathbf{R} \times \mathbf{F}_{\mathrm{ext}} + \sum_a \mathbf{r}'_a \times \mathbf{F}_{a,\mathrm{ext}}. \tag{7.10}$$

The separate components have their respective torques,

$$\dot{\mathbf{L}}_{\mathrm{CM}} = \mathbf{R} \times \mathbf{F}_{\mathrm{ext}} = \boldsymbol{\tau}_{\mathrm{ext}}, \quad \dot{\mathbf{L}}_{\mathrm{rel}} = \sum_a \mathbf{r}'_a \times \mathbf{F}_{a,\mathrm{ext}} = \boldsymbol{\tau}'_{\mathrm{ext}}. \tag{7.11}$$

These statements about the cancellation of internal forces and torques follow from the general action principle statement,

$$\delta W = G_1 - G_2, \quad G = \sum_a \mathbf{p}_a \cdot \delta\mathbf{r}_a. \tag{7.12}$$

If the external force and torque are zero, the system is invariant under either a rigid spatial translation or a rotation, so

$$G = \sum_a \mathbf{p}_a \cdot \delta\mathbf{r}_a = \delta\boldsymbol{\epsilon} \cdot \sum_a \mathbf{p}_a = \delta\boldsymbol{\epsilon} \cdot \mathbf{P}, \tag{7.13a}$$

$$G = \sum_a \mathbf{p}_a \cdot \delta\boldsymbol{\omega} \times \mathbf{r}_a = \delta\boldsymbol{\omega} \cdot \sum_a \mathbf{r}_a \times \mathbf{p}_a = \delta\boldsymbol{\omega} \cdot \mathbf{L}, \tag{7.13b}$$

which says that $\mathbf{P}$ and $\mathbf{L}$ are constant, meaning that the internal force changes neither the linear momentum nor the angular momentum.

The kinetic energy has a similar breakup,

$$\begin{aligned} T &= \frac{1}{2}\sum_a m_a \dot{\mathbf{r}}_a^2 = \frac{1}{2}\sum_a m_a(\dot{\mathbf{R}} + \dot{\mathbf{r}}'_a)^2 \\ &= \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\sum_a m_a \dot{\mathbf{r}}_a'^2 = T_{\mathrm{CM}} + T_{\mathrm{rel}}. \end{aligned} \tag{7.14}$$

Now we specialize to a rigid body, which can only undergo rigid rotations about its center of mass. If the instantaneous angular velocity vector is $\boldsymbol{\Omega}$, this means the relative velocity of the ath particle is

$$\mathbf{v}'_a = \boldsymbol{\Omega} \times \mathbf{r}'_a. \tag{7.15}$$

The relative kinetic energy is then

$$T_{\mathrm{rel}} = \frac{1}{2}\sum_a m_a(\boldsymbol{\Omega} \times \mathbf{r}'_a)^2 \tag{7.16}$$

$$\begin{aligned} &= \frac{1}{2}\sum_a m_a \boldsymbol{\Omega} \cdot \mathbf{r}'_a \times (\boldsymbol{\Omega} \times \mathbf{r}'_a) = \frac{1}{2}\sum_a m_a \boldsymbol{\Omega} \cdot (\boldsymbol{\Omega} r_a'^2 - \mathbf{r}'_a(\mathbf{r}'_a \cdot \boldsymbol{\Omega})) \\ &= \frac{1}{2}\boldsymbol{\Omega} \cdot \mathbf{I} \cdot \boldsymbol{\Omega}, \end{aligned} \tag{7.17}$$

where we have introduced the moment of inertia tensor,

$$\mathbf{I} = \sum_a m_a(r_a'^2 \mathbf{1} - \mathbf{r}'_a \mathbf{r}'_a), \tag{7.18}$$

or

$$I_{ij} = \sum_a m_a(r_a'^2 \delta_{ij} - r'_{ai} r'_{aj}). \tag{7.19}$$

The moment of inertia tensor also occurs in the relative angular momentum,

$$\mathbf{L}_{\mathrm{rel}} = \sum_a m_a \mathbf{r}'_a \times (\boldsymbol{\Omega} \times \mathbf{r}'_a) = \sum_a m_a(r_a'^2 \boldsymbol{\Omega} - (\mathbf{r}'_a \cdot \boldsymbol{\Omega})\mathbf{r}'_a), \tag{7.20}$$

or

$$\mathbf{L}_{\text{rel}} = \mathbf{I} \cdot \mathbf{\Omega}. \tag{7.21}$$

Note that the moment of inertia tensor is symmetric,

$$\mathbf{I} = \mathbf{I}^T, \quad \text{or} \quad I_{ij} = I_{ji}, \tag{7.22}$$

and of course it is real. Therefore, there is an orthogonal basis, attached to the body, in which $\mathbf{I}$ is diagonal. The directions of the basis vectors are called the *principal axes*. In the principal axis system,

$$I_{ij} = I_i \delta_{ij}. \tag{7.23}$$

In the principal axis system

$$T_{\text{rot}} = \frac{1}{2}(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2), \tag{7.24}$$

and

$$\mathbf{L}_{\text{rel}} = (I_1\Omega_1, I_2\Omega_2, I_3\Omega_3). \tag{7.25}$$

Let the coordinates in the principal axis system be x, y, z. Then

$$I_x = \sum_a m_a(y_a^2 + z_a^2), \quad I_y = \sum_a m_a(z_a^2 + x_a^2), \quad I_z = \sum_a m_a(x_a^2 + y_a^2), \tag{7.26}$$

so we have the necessary inequality

$$I_x + I_y > I_z, \tag{7.27}$$

and so on by permutations.

The moment of inertia tensor reflects symmetries of the body. If in the principal axis system $I_1 = I_2 = I_3$, we call the body a spherical top; in that case, the body has spherical symmetry and any set of Cartesian axes can serve as principal axes. In that case $\mathbf{I} = I\mathbf{1}$ and so $\mathbf{I}$ is a constant, so then, and in no other case, $\dot{\mathbf{L}}_{\text{rel}} = I\dot{\mathbf{\Omega}}$.

An asymmetric top has $I_1 \neq I_2 \neq I_3$. A symmetric top has $I_1 = I_2 \neq I_3$. Let us consider the free (no external forces) motion of a symmetric top. Then $\mathbf{L} = \mathbf{I} \cdot \mathbf{\Omega}$ is a constant. As illustrated in Fig. 7.2, the symmetry axis of the body is the z axis, and consider the plane defined by that axis and the constant $\mathbf{L}$. Choose the y axis to be perpendicular to that plane. Then because $L_y = 0$, $\Omega_y = 0$, so $\mathbf{\Omega}$ also lies in the plane defined by z and $\mathbf{L}$. If $\mathbf{r}$ represents a point on the symmetry axis, its velocity is $\mathbf{v} = \mathbf{\Omega} \times \mathbf{r}$ which is perpendicular to the z-$\mathbf{L}$ plane, so the symmetry axis of the body precesses uniformly about the fixed $\mathbf{L}$ direction; at the same time, the body rotates about the symmetry axis. Let the angle between $\mathbf{L}$ and $\hat{\mathbf{z}}$ be θ. Then the angular velocity of the top rotating about its symmetry axis is

$$\Omega_z = \frac{L_z}{I_z} = \frac{L}{I_3}\cos\theta. \tag{7.28}$$

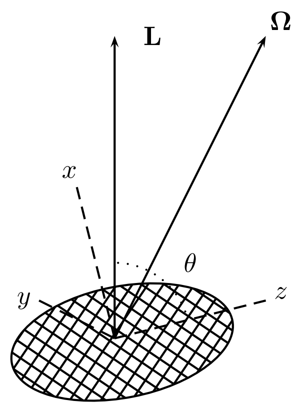

Figure 7.2: A free symmetric top with symmetry axis along the z axis. The x, y, z axes shown are the principal body axis system. The conserved angular momentum L and the z body axis define a plane, while the y axis is chosen to be perpendicular to that plane, and the x axis lies in that plane, both of which are perpendicular to z, of course. Let the angle between z and $\mathbf{L}$ be θ. Because $L_y = I_2\Omega_2 = 0$, the angular velocity $\mathbf{\Omega}$ lies in the x-y plane. The result is that the symmetry axis z precesses uniformly about the direction of $\mathbf{L}$, maintaining a constant angle θ with respect to that axis, while simultaneously the body rotates (spins) about the symmetry axis z.

The precessional velocity Ω_p is related to the component of $\mathbf{\Omega}$ in the x direction (which lies in the z-$\mathbf{L}$ plane) by

$$\Omega_x = \Omega_p \sin\theta = \frac{L_1}{I_1} = \frac{L\sin\theta}{I_1}, \tag{7.29}$$

so the precessional velocity is

$$\Omega_p = \frac{L}{I_1}. \tag{7.30}$$

7.1 Euler Angles

Now we have to discuss how the body principal axis system rotates relative to an inertial system, or a system with origin at the center of mass of the body, but with axes parallel to an inertial system of coordinate axes. Conventionally, this is done in terms of three Euler angles. These are illustrated in Fig. 7.3. We start with the CM system with axes parallel to those in an inertial frame, where the coordinates are designated (x, y, z). First we rotate through an angle ϕ about the

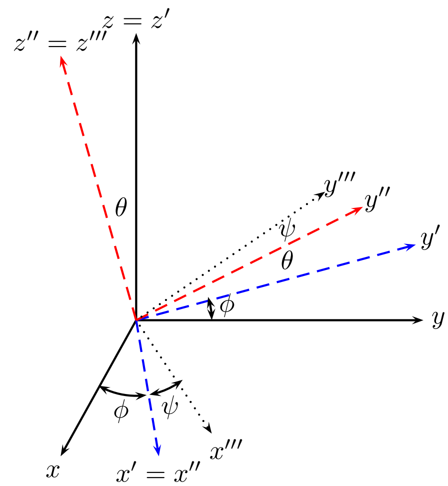

Figure 7.3: The three Euler angles describing the transition between a coordinate system with axes parallel to an inertial system of Cartesian coordinates, and the body principal axis system. The former coordinates are designated (x, y, z) and the latter (x''', y''', z'''). The meaning of the Euler angles (ϕ, θ, ψ) is given in the text.

z axis, going to a coordinate system with $(x', y', z' = z)$. Second, we rotate about the x' axis through an angle θ giving us coordinates $(x'' = x', y'', z'')$. Finally we rotate through an angle ψ about the z'' axis, carrying us to the principal axis system, $(x''', y''', z''' = z'')$. In this way we can reach any desired orientation of the body. [In quantum mechanics, the conventional second rotation is about the y' axis.] The line $x' = x''$ is called the *line of nodes.*

Each of these successive rotations is a simple rotation about a single axis, and can be described by rotation matrices:

$$R_1 = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}, \quad R_3 = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{7.31}$$

and so the net rotation is given by the product of the matrices, taken from right to left:

$$\begin{aligned} R_3R_2R_1 &= \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos\phi\cos\psi - \sin\phi\cos\theta\sin\psi & \sin\phi\cos\psi + \cos\phi\cos\theta\sin\psi & \sin\theta\sin\psi \\ -\cos\phi\sin\psi - \sin\phi\cos\theta\cos\psi & -\sin\phi\sin\psi + \cos\phi\cos\theta\cos\psi & \sin\theta\cos\psi \\ \sin\phi\sin\theta & -\cos\phi\sin\theta & \cos\theta \end{pmatrix}. \end{aligned} \tag{7.32}$$

Using these we can rotate an arbitrary angular velocity to the body system. The velocity corresponding to a ϕ rotation only in the original coordinate system is

$$\boldsymbol{\Omega}_\phi = \begin{pmatrix} 0 \\ 0 \\ \dot\phi \end{pmatrix}, \tag{7.33}$$

which turns into, in the body system

$$\bar{\boldsymbol{\Omega}}_\phi = R_3 R_2 R_1 \boldsymbol{\Omega}_\phi = \begin{pmatrix} \dot\phi \sin\theta \sin\psi \\ \dot\phi \sin\theta \cos\psi \\ \dot\phi \cos\theta \end{pmatrix}. \tag{7.34}$$

A velocity corresponding to a θ rotation only requires only one subsequent rotation,

$$\bar{\boldsymbol{\Omega}}_\theta = R_3 \boldsymbol{\Omega}_\theta = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot\theta \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \dot\theta \cos\psi \\ -\dot\theta \sin\psi \\ 0 \end{pmatrix}. \tag{7.35}$$

A velocity corresponding to a ψ rotation only requires no subsequent rotation

$$\bar{\boldsymbol{\Omega}}_\psi = \boldsymbol{\Omega}_\psi = \begin{pmatrix} 0 \\ 0 \\ \dot\psi \end{pmatrix}. \tag{7.36}$$

Thus a general angular velocity in the body principal axis system is the sum of these three components:

$$\bar{\boldsymbol{\Omega}} = \bar{\boldsymbol{\Omega}}_\phi + \bar{\boldsymbol{\Omega}}_\theta + \bar{\boldsymbol{\Omega}}_\psi = \begin{pmatrix} \dot\theta \cos\psi + \dot\phi \sin\theta \sin\psi \\ -\dot\theta \sin\psi + \dot\phi \sin\theta \cos\psi \\ \dot\phi \cos\theta + \dot\psi \end{pmatrix}. \tag{7.37}$$

This result can also be verified geometrically.

Let us consider the example of the free symmetric top again, where $I_1 = I_2 \neq I_3$. Then, the kinetic energy of the top is

$$T = \frac{1}{2} I_1 (\dot\theta^2 + \dot\phi^2 \sin^2\theta) + \frac{1}{2} I_3 (\dot\phi \cos\theta + \dot\psi)^2. \tag{7.38}$$

To describe free motion, $\dot{\mathbf{L}} = 0$, let us choose the fixed direction of $\mathbf{L}$ to coincide with the fixed z axis. In this case we can, without loss of generality, assume $\psi = 0$ at one instant, but, of course, not $\dot\psi = 0$. Then in the principal axis system, $\Omega_1 = \dot\theta$, $\Omega_2 = \dot\phi \sin\theta$, $\Omega_3 = \dot\phi \cos\theta + \dot\psi$. Our choice of $\psi = 0$ means that the principal axis x''' is the line of nodes and is perpendicular to $\mathbf{L}$. Thus (subscripts refer to principal axes) $L_1 = I_1 \Omega_1 = I_1 \dot\theta = 0$, or $\dot\theta = 0$. That is, the angle between the symmetry axis and the direction of the angular momentum, θ, is a constant. The second principal axis component of $\mathbf{L}$ is

$$L_2 = I_2 \Omega_2 = I_2 \dot\phi \sin\theta = L \sin\theta, \tag{7.39}$$

so $\dot\phi = L/I_2$, as we found before. This is the precessional velocity of the top. Finally, the third principal axis component is

$$L_3 = I_3\Omega_3 = L\cos\theta, \tag{7.40}$$

or $\Omega_3 = (L/I_3)\cos\theta$, which is the spin velocity, the rotational velocity of the body about its symmetry axis.

7.2 Euler's equations

Now we want to relate the motion of the body in the body system to that in the fixed system of coordinates. Any vector $\mathbf{A}$ changes both due to its infinitesimal change in the body system, $\delta'\mathbf{A}$ and the change that a vector makes under a rotation:

$$\delta\mathbf{A} = \delta'\mathbf{A} + \delta\boldsymbol{\omega}\times\mathbf{A}, \tag{7.41}$$

or, dividing by δt, we see how the time derivative in the fixed system is composed of two parts,

$$\frac{d\mathbf{A}}{dt} = \frac{d'\mathbf{A}}{dt} + \boldsymbol{\Omega}\times\mathbf{A}, \quad \boldsymbol{\Omega} = \frac{d\boldsymbol{\omega}}{dt}. \tag{7.42}$$

Thus, the equation of motion of the center of mass becomes in the body system

$$\frac{d\mathbf{P}}{dt} = \mathbf{F} = \frac{d'\mathbf{P}}{dt} + \boldsymbol{\Omega}\times\mathbf{P}, \tag{7.43}$$

which means in the body system

$$F_1 = \frac{d'P_1}{dt} + \Omega_2 P_3 - \Omega_3 P_2 = M\left(\frac{d'V_1}{dt} + \Omega_2 V_3 - \Omega_3 V_2\right), \tag{7.44}$$

and so on by cyclic permutations.

The angular momentum equation of motion is

$$\frac{d\mathbf{L}}{dt} = \boldsymbol{\tau} = \frac{d'\mathbf{L}}{dt} + \boldsymbol{\Omega}\times\mathbf{L}, \tag{7.45}$$

in the body system. Now in the body system

$$\mathbf{L} = (L_1, L_2, L_3) \tag{7.46}$$

so

$$\tau_1 = I_1\dot\Omega_1 + \Omega_2 I_3\Omega_3 - \Omega_3 I_2\Omega_2 = I_1\dot\Omega_1 + (I_3 - I_2)\Omega_2\Omega_3, \tag{7.47}$$

and so on by cyclic permutations. These equations are called Euler's equations. In the following we will drop the prime for derivatives in the body (rotating) system.

Now for the third time let us revisit the free rotation of a symmetric top, where $I_1 = I_2 \neq I_3$ and $\boldsymbol{\tau} = 0$. The Euler equations then read

$$\frac{d\Omega_1}{dt} = \frac{I_1 - I_3}{I_1}\Omega_2\Omega_3, \quad \frac{d\Omega_2}{dt} = -\frac{I_1 - I_3}{I_1}\Omega_1\Omega_3, \quad \frac{d\Omega_3}{dt} = 0. \tag{7.48}$$

Thus Ω_3 is constant, and we can define the frequency

$$\omega = \Omega_3 \frac{I_3 - I_1}{I_1}, \tag{7.49}$$

so the equations for Ω_1, Ω_2 are

$$\dot\Omega_1 = -\omega\Omega_2, \quad \dot\Omega_2 = \omega\Omega_1, \tag{7.50}$$

which implies

$$\ddot\Omega_1 = -\omega^2\Omega_1. \tag{7.51}$$

So with a choice of initial time, the solution is

$$\Omega_1 = A\cos\omega t, \quad \Omega_2 = A\sin\omega t. \tag{7.52}$$

This says that the magnitude A of the transverse ($\perp$ to symmetry axis) angular velocity is constant, and that the angular velocity precesses with angular velocity $\boldsymbol{\omega} = \hat{\mathbf{z}}'''\omega$ about the symmetry axis of the top. Likewise the angular momentum precesses about the axis of the top with the same velocity.

This result, of course, must agree with that found previously. We refer to the Euler angles. We note that that the angular velocity of $\mathbf{L}$ about the symmetry axis, the z''' axis, is $-\dot\psi$. Now the angular momentum along that direction is $L_3 = L\cos\theta = I_3\Omega_3$, and we recall from Eq. (7.37) that $\Omega_3 = \dot\phi\cos\theta + \dot\psi$, and so

$$\dot\psi = -\dot\phi\cos\theta + \Omega_3 = -\frac{L}{I_2}\cos\theta + \Omega_3 = \frac{I_1 - I_3}{I_1}\Omega_3, \tag{7.53}$$

so $\omega_p = -\dot\psi$ is the same precessional angular velocity found in Eq. (7.49).

7.3 Free motion of an asymmetric top

Without loss of generality now assume $I_3 > I_2 > I_1$. We first prove that rotations about the 1 or the 3 axis is stable, but not about the one possessing the intermediate moment of inertia. This is rather easily seen from the Euler equations,

$$I_1\dot\Omega_1 + (I_3 - I_2)\Omega_2\Omega_3 = 0, \tag{7.54a}$$
$$I_2\dot\Omega_2 + (I_1 - I_3)\Omega_3\Omega_1 = 0, \tag{7.54b}$$
$$I_3\dot\Omega_3 + (I_2 - I_1)\Omega_1\Omega_2 = 0, \tag{7.54c}$$

because the second difference of moments of inertia is negative, while the others are positive. Suppose the the initial motion is about the 1 axis,

$$\boldsymbol{\Omega}^{(0)} = (\Omega^{(0)}, 0, 0). \tag{7.55}$$

which is consistent with the Euler equations if $\Omega^{(0)}$ is constant. But now suppose there is a small deviation away from $\boldsymbol{\Omega}^{(0)}$, say

$$\boldsymbol{\Omega} = \boldsymbol{\Omega}^{(0)} + \epsilon\boldsymbol{\Omega}^{(1)}, \tag{7.56}$$

where ϵ is a small parameter. Then, to first order in ϵ, the deviations in the angular velocity satisfy

$$\begin{aligned} I_1 \frac{d\Omega_1^{(1)}}{dt} &= 0, && (7.57a)\\ I_2 \frac{d\Omega_2^{(1)}}{dt} + (I_1 - I_3)\Omega_1^{(0)}\Omega_3^{(1)} &= 0, && (7.57b)\\ I_3 \frac{d\Omega_3^{(1)}}{dt} + (I_2 - I_1)\Omega_1^{(0)}\Omega_2^{(1)} &= 0, && (7.57c) \end{aligned}$$

and then the last two equations imply

$$\frac{d^2\Omega_2^{(1)}}{dt^2} + \omega^2 \Omega_2^{(1)} = 0, \tag{7.58}$$

where

$$\omega^2 = \frac{(I_1 - I_3)(I_1 - I_2)}{I_2 I_3} \Omega_1^{(0)2} \tag{7.59}$$

is positive. This means that the transverse components of the angular velocity undergo circular rotation about the 1 axis, and that the rotation about the 1 axis is stable. The same conclusion holds for rotation about the 3 axis (see homework). However, if the initial angular velocity is about the 2 axis,

$$\boldsymbol{\Omega}^{(0)} = (0, \Omega^{(0)}, 0), \tag{7.60}$$

and then we consider a small perturbation as in Eq. (7.56), the Euler's equation for the perturbations are

$$\begin{aligned} I_2 \frac{d\Omega_2^{(1)}}{dt} &= 0, && (7.61a)\\ I_1 \frac{d\Omega_1^{(1)}}{dt} + (I_3 - I_2)\Omega_2^{(0)}\Omega_3^{(1)} &= 0, && (7.61b)\\ I_3 \frac{d\Omega_3^{(1)}}{dt} + (I_2 - I_1)\Omega_2^{(0)}\Omega_1^{(1)} &= 0, && (7.61c) \end{aligned}$$

and then the last two equations are combined to read

$$\frac{d^2\Omega_3^{(1)}}{dt^2} - \kappa^2 \Omega_3^{(1)} = 0, \quad \kappa^2 = -\frac{(I_2 - I_1)(I_2 - I_3)}{I_1 I_3} \Omega_2^{(0)}, \tag{7.62}$$

where κ^2 is positive. Now the solutions are exponential,

$$\Omega_3^{(1)} \propto e^{\pm \kappa t}. \tag{7.63}$$

The exponentially growing solution means that the perturbation will grow rapidly, and the original solution is unstable.

7.4 General motion of a free asymmetric top

As we have seen many times previously, the Euler equation can be reduced to quadratures. There are two constants of the motion, the energy and the magnitude of the angular momentum,

$$E = \frac{1}{2}(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2), \tag{7.64a}$$

$$L^2 = I_1^2\Omega_1^2 + I_2^2\Omega_2^2 + I_3^2\Omega_2^3. \tag{7.64b}$$

It is useful to think in three-dimensional angular momentum space. The first equation here says that the angular momentum vector connects the center of mass and the surface of an ellipsoid around that point, with semiaxes

$$(\sqrt{2EI_1}, \sqrt{2EI_2}, \sqrt{2EI_3}). \tag{7.65}$$

The second says that the angular momentum connects the center of mass to the surface of a surrounding sphere with radius L. Thus $\mathbf{L}$ moves relative to the body system on the intersection of these two surfaces. That these two surfaces intersect follows from (we are still assuming $I_1 < I_2 < I_3$)

$$2I_1E = L_1^2 + \frac{I_1}{I_2}L_2^2 + \frac{I_1}{I_3}L_3^2 < L_1^2 + L_2^2 + L_3^2 = L^2 < L_3^2 + \frac{I_3}{I_2}L_2^2 + \frac{I_3}{I_1}L_1^2 = 2I_3E, \tag{7.66}$$

or

$$2I_1E < L^2 < 2I_3E, \tag{7.67}$$

which says that the radius of the sphere is intermediate between the largest and smallest semiaxes of the momentum ellipsoid, so there are curves of intersection.

Now we can easily solve the energy and momentum equations (7.64a), (7.64b) for Ω_1^2 and Ω_3^2:

$$\Omega_3^2 = \frac{2EI_1 - L^2 - I_2(I_1 - I_2)\Omega_2^2}{I_3(I_1 - I_3)}, \tag{7.68a}$$

$$\Omega_1^2 = \frac{2EI_3 - L^2 - I_2(I_3 - I_2)\Omega_2^2}{I_1(I_3 - I_1)}, \tag{7.68b}$$

and when this is substituted into the 2nd Euler equation, we get,

$$\frac{d\Omega_2}{dt} = \frac{1}{I_2\sqrt{I_1I_3}}\sqrt{[L^2 - 2EI_1 - I_2(I_2 - I_1)\Omega_2^2][2EI_3 - L^2 - I_2(I_3 - I_2)\Omega_2^2]}. \tag{7.69}$$

Making the following simple changes of variable,

$$\tau = t\sqrt{\frac{(I_3 - I_2)(L^2 - 2EI_1)}{I_1I_2I_3}}, \quad s = \Omega_2\sqrt{\frac{I_2(I_3 - I_2)}{2EI_3 - L^2}}, \tag{7.70}$$

we have

$$\frac{ds}{d\tau} = \sqrt{(1 - s^2)(1 - k^2s^2)}, \tag{7.71}$$

where

$$k^2 = \frac{(I_2 - I_1)(2EI_3 - L^2)}{(I_3 - I_2)(L^2 - 2EI_1)}. \tag{7.72}$$

Here we assume that $L^2 > 2EI_2$ so that $0 < k^2 < 1$. The turning points occur at $s = \pm 1$, therefore, by symmetry, the time required to go from $s = 0$ to $s = 1$ is a quarter period of a complete motion, say from $s = -1$ to $s = +1$ and back again.

As should be familiar by now, we can write the time in terms of an integral,

$$\tau = \int_0^s \frac{ds'}{\sqrt{(1 - s'^2)(1 - k^2 s'^2)}} = F(\arcsin s | k^2), \quad s \leq 1. \tag{7.73}$$

Here F is the elliptic integral of the first kind,

$$F(\phi | m) = \int_0^\phi d\theta (1 - m \sin^2\theta)^{-1/2}. \tag{7.74}$$

When $s = 1$ the angular velocity has executed a quarter period, so the period in τ is then given in terms of the complete elliptic integral K,

$$\tilde{T} = 4K, \quad K = \int_0^1 \frac{ds'}{\sqrt{(1 - s'^2)(1 - k^2 s'^2)}} = F(\pi/2 | k^2) \equiv K(k^2), \tag{7.75}$$

and then the physical period is given by

$$T = 4K(k^2) \sqrt{\frac{I_1 I_2 I_3}{(I_3 - I_2)(L^2 - 2EI_1)}}. \tag{7.76}$$

However, although the angular momentum precesses about the body symmetry axis with this period, the top does not rotate with this period relative to the inertial system, but with a motion that also involves an incommensurate period; see Landau and Lifshitz for further discussion. The top never returns exactly to its original position!

7.5 Problems for Chapter 7

1. Discuss the motion of the free asymmetric top when $2EI_2 > L^2$.
2. Prove that the formula for the period of the asymmetric top can be recast in the form of a complete elliptic integral as in Eq. (7.76).