Chapter 4

Relativistic Dynamics

The most important example of a relativistic particle moving in a potential is a charged particle, say an electron, moving in an electromagnetic field, which might be that of a nucleus. In the absence of magnetic charge (magnetic monopoles) the two Maxwell equations that do not refer to electrically charged particles are

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0, \quad -\boldsymbol{\nabla} \times \mathbf{E} = \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}. \tag{4.1}$$

The first of these implies that $\mathbf{B}$ can be constructed from a vector potential,

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}, \tag{4.2}$$

while the second can then be rearranged to

$$\boldsymbol{\nabla} \times \left(\mathbf{E} + \frac{1}{c}\frac{\partial}{\partial t}\mathbf{A} \right) = 0, \tag{4.3}$$

which implies the existence of a scalar potential,

$$\mathbf{E} = -\frac{1}{c}\frac{\partial}{\partial t}\mathbf{A} - \boldsymbol{\nabla}\phi. \tag{4.4}$$

We can unite the scalar and vector potential into a single 4-vector, A^μ, $A^0 = \phi$, $A_i = (\mathbf{A})_i$. At this point, we simply assert (it will be proved in the Electrodynamics course) that A^μ transforms as a 4-vector. A covariant way of writing the construction of the electric and magnetic fields in terms of the potentials is in terms of the field-strength tensor ($\partial^\mu = \partial/\partial x_\mu$)

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \tag{4.5}$$

which is evidently antisymmetric, $F^{\mu\nu} = -F^{\nu\mu}$, and has 6 independent components:

$$F^{0i} = E_i, \quad F^{ij} = \epsilon_{ijk} B_k \quad (\text{so} \quad F^{12} = B_3, \quad \text{etc.}). \tag{4.6}$$

In terms of these potentials, the action describing the interaction of a particle with rest mass m_0 and charge e has the following Lorentz invariant form

$$W_{12} = \int_2^1 \left(-m_0 c^2 d\tau + \frac{e}{c} A_\mu dx^\mu\right). \tag{4.7}$$

Recalling that $cd\tau = \sqrt{-dx_\mu dx^\mu}$, we obtain upon varying the action

$$\begin{aligned}
\delta W_{12} &= -m_0 c \int_2^1 (-dx_\mu dx^\mu)^{-1/2} \left(-dx^\lambda d\delta x_\lambda\right) \\
&\quad + \frac{e}{c} \int_2^1 \left(A_\mu d\delta x^\mu + \delta x^\lambda \partial_\lambda A_\mu dx^\mu\right) \\
&= m_0 \int_2^1 \left[d\left(\delta x_\mu \frac{dx^\mu}{d\tau}\right) - \delta x_\mu \frac{d^2 x^\mu}{d\tau}\right] \\
&\quad + \frac{e}{c} \int_2^1 \left[d(\delta x_\mu A^\mu) - \delta x^\mu dA_\mu + \delta x^\lambda \partial_\lambda A_\mu dx^\mu\right] \\
&= \delta x_\mu \left(m_0 \frac{dx^\mu}{d\tau} + \frac{e}{c} A^\mu\right)\Bigg|_2^1 \\
&\quad + \int_2^1 d\tau \delta x_\mu \left[-m_0 \frac{d^2 x^\mu}{d\tau^2} - \frac{e}{c}\frac{dA^\mu}{d\tau} + \frac{e}{c} \partial^\mu A_\lambda \frac{dx^\lambda}{d\tau}\right].
\end{aligned} \tag{4.8}$$

From this, we may read off the generators,

$$G = p^\mu \delta x_\mu, \quad p^\mu = m_0 \frac{dx^\mu}{d\tau} + \frac{e}{c} A^\mu, \tag{4.9}$$

which gives the canonical momentum in terms of the mechanical momentum and the vector potential, and the equation of motion,

$$\frac{d}{d\tau} p^\mu = \frac{e}{c} \partial^\mu A_\lambda \frac{dx^\lambda}{d\tau}. \tag{4.10}$$

The last equation gives the energy nonconservation equation and the Lorentz force law: We can immediately rewrite it as

$$m_0 \frac{d^2 x^\mu}{d\tau^2} = \frac{e}{c} \left(\partial^\mu A^\lambda - \partial^\lambda A^\mu\right) \frac{dx_\lambda}{d\tau} = \frac{e}{c} F^{\mu\lambda} \frac{dx_\lambda}{d\tau}. \tag{4.11}$$

The left-hand side of the $\mu = 0$ component of this equation reads

$$m_0 \frac{d^2 x^0}{d\tau^2} = m_0 c \frac{d}{d\tau}\gamma = \frac{d}{d\tau}\frac{E}{c} = \gamma \frac{d}{dt}\frac{E}{c}, \tag{4.12}$$

while the right-hand side of the same component equation is

$$\frac{e}{c} F^{0i} \frac{dx^i}{d\tau} = \frac{e}{c} \gamma \mathbf{E} \cdot \mathbf{v}, \quad \mathbf{v} = \frac{d\mathbf{x}}{dt}, \tag{4.13}$$

so combining these we see

$$\frac{d}{dt}E = e\mathbf{E}\cdot\mathbf{v}, \tag{4.14}$$

where $E = m_0\gamma c^2$ is the mechanical energy of the particle; only the electric field does work on the charged particle.

In terms of the mechanical momentum of the charged particle $\mathbf{p} = m_0\gamma\mathbf{v}$, the $\mu = i$ component of Eq. (4.11) is

$$m_0\frac{d^2x_i}{d\tau^2} = \frac{d}{d\tau}\mathbf{p}_i = \frac{e}{c}\left(F^{ij}\frac{dx_j}{d\tau} + F^{i0}\frac{dx_0}{d\tau}\right) = e\gamma\left(\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B}\right)_i, \tag{4.15}$$

or

$$\frac{d}{dt}\mathbf{p} = e\left(\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B}\right). \tag{4.16}$$

This is the Lorentz force law.

Finally, it is easy to check that taking the dot product of this equation with $\mathbf{v}$, which gives the power, yields the energy nonconservation equation (4.14).

4.1 Motion in a uniform magnetic field

Consider now the motion of a charged particle in a uniform magnetic field, so the equations of motion read (now SI units)

$$\frac{d\mathbf{p}}{dt} = e\mathbf{v}\times\mathbf{B}, \quad \mathbf{p} = m_0\gamma\mathbf{v} = \frac{E}{c^2}\mathbf{v}, \quad \frac{dE}{dt} = 0. \tag{4.17}$$

This then means

$$\frac{E}{c^2}\frac{d\mathbf{v}}{dt} = e\mathbf{v}\times\mathbf{B}, \tag{4.18}$$

or

$$\frac{d\mathbf{v}}{dt} = \boldsymbol{\omega}\times\mathbf{v}, \quad \boldsymbol{\omega} = -\frac{ec^2}{E}\mathbf{B}, \tag{4.19}$$

which says that the velocity vector $\mathbf{v}$ precesses about the direction of the magnetic field with angular velocity $\boldsymbol{\omega}$. If we confine the motion to the x-y plane, this says the particle moves in a circular orbit in that plane,

$$v_x(t) = v_x(0)\cos\omega t + v_y(0)\sin\omega t, \tag{4.20a}$$

$$v_y(t) = v_y(0)\cos\omega t - v_x(0)\sin\omega t, \tag{4.20b}$$

with $\omega = ec^2B/E$. The speed of the particle is related to the angular speed by $v = \omega R$, so radius of the orbit is

$$R = \frac{\beta E}{|e|cB}. \tag{4.21}$$

For example, the proton synchrotron at the Large Hadron Collider (LHC) has a circumference of 4.25 km, and an ultimate beam energy of 7 TeV. The constant vertical magnetic field required to bend the protons into a circular orbit would be

$$B = \frac{7\,\mathrm{TeV}}{e\,3\times 10^8\,\mathrm{m/s}\,4.25\times 10^3\mathrm{m}} = 5.5\,\mathrm{T}, \tag{4.22}$$

but since the bending magnets are not providing a uniform field over the entire orbit, they necessarily produce a higher peak magnetic field, about 8.3 T.

Note that the radius of curvature of a charged particle in a magnetic field provides a practical way to measure momentum of such particles:

$$p = \frac{E}{c^2}v = \frac{E\beta}{c} = eBR. \tag{4.23}$$

4.2 Relativistic orbits

Now let us consider a relativistic orbit problem with a $-\alpha/r$ potential. That is, we take as the Lagrangian

$$L = -m_0c^2\sqrt{1-\beta^2} + \frac{\alpha}{r}. \tag{4.24}$$

The Hamiltonian is

$$H = \sqrt{p^2c^2 + m_0^2c^4} - \frac{\alpha}{r}. \tag{4.25}$$

The orbit is confined to a plane, in which we adopt polar coordinates,

$$x = r\cos\theta, \quad y = r\sin\theta, \tag{4.26a}$$

$$\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta, \quad \dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta. \tag{4.26b}$$

Thus the square of the velocity is

$$v^2 = \dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2\dot{\theta}^2, \tag{4.27}$$

so the Lagrangian is

$$L = -m_0c^2\sqrt{1 - \frac{1}{c^2}(\dot{r}^2 + r^2\dot{\theta}^2)} + \frac{\alpha}{r}. \tag{4.28}$$

The canonical momenta are

$$p_\theta = \frac{\partial L}{\partial\dot{\theta}} = m_0\gamma r^2\dot{\theta} = \ell, \tag{4.29a}$$

$$p_r = \frac{\partial L}{\partial\dot{r}} = m_0\gamma\dot{r}. \tag{4.29b}$$

Since θ does not occur in the Lagrangian, p_θ is a constant of the motion, and coincides with the angular momentum in the plane,

$$\ell = m_0r\gamma v_\theta, \quad v_\theta = r\dot{\theta}. \tag{4.30}$$

Now notice that

$$\frac{p_r}{p_\theta} = \frac{\dot{r}}{r^2\dot{\theta}} = \frac{1}{r^2}\frac{dr}{d\theta}, \tag{4.31}$$

so

$$p_r = \frac{1}{r^2}\frac{dr}{d\theta}\ell. \tag{4.32}$$

The energy equation can be written as

$$\begin{aligned}\left(E + \frac{\alpha}{r}\right)^2 &= p^2c^2 + m_0^2c^4 = \left(p_r^2 + \frac{p_\theta^2}{r^2}\right)c^2 + m_0^2c^4 \\ &= \frac{c^2\ell^2}{r^4}\left[\left(\frac{dr}{d\theta}\right)^2 + r^2\right] + m_0^2c^4. \end{aligned}\tag{4.33}$$

As we did nonrelativistically, let $r = 1/s$, so

$$\frac{dr}{d\theta} = -\frac{1}{s^2}\frac{ds}{d\theta}, \tag{4.34}$$

and then the energy equation reads

$$(E + \alpha s)^2 - m_0^2c^4 = c^2\ell^2\left(\frac{ds}{d\theta}\right)^2 + c^2\ell^2s^2. \tag{4.35}$$

This can be turned into the differential statement

$$d\theta = \frac{ds}{\sqrt{\frac{(E+\alpha s)^2 - m_0^2c^4}{c^2\ell^2} - s^2}}. \tag{4.36}$$

By completing the square, and introducing some abbreviations,

$$\hat{\alpha} = \frac{\alpha}{c\ell}, \quad \hat{E} = \frac{E}{c\ell}, \quad \hat{E}_0 = \frac{m_0c^2}{c\ell}, \tag{4.37}$$

the differential equation is equivalent to

$$d\theta = \frac{ds}{\sqrt{(\hat{\alpha}^2 - 1)\left(s + \frac{\hat{\alpha}\hat{E}}{\hat{\alpha}^2-1}\right)^2 - \frac{\hat{E}^2}{\hat{\alpha}^2-1} - \hat{E}_0^2}}. \tag{4.38}$$

We will suppose $\hat{\alpha}^2 < 1$; for the case of the Coulomb potential, this says $Ze^2/c = Z\hbar\alpha_0 < \ell$, which is not much of a restriction since α_0, the fine structure constant, has the value $1/137$. If we further suppose that

$$\frac{\hat{E}^2}{1-\alpha^2} - \hat{E}_0^2 > 0, \tag{4.39}$$

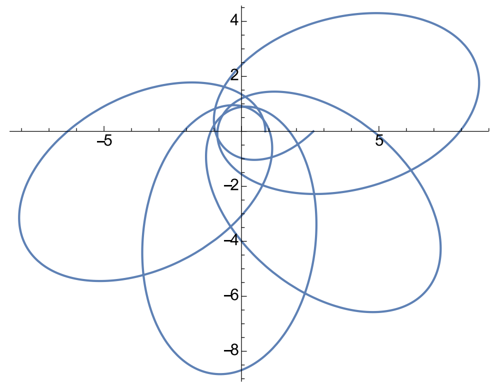

Figure 4.1: Bound relativistic orbit in a $-\alpha/r$ potential. The parameters are $\hat{\alpha} = 0.5$, $\hat{E} = 0.95$, $\hat{E}_0 = 1$. Note that the orbits do not close, but precess about the origin. Shown are 5 encirclings of the origin.

the differential equation (4.38) may be immediately integrated as an arcsine, with the result

$$\begin{aligned} \frac{1}{r} &= \frac{\hat{\alpha}\hat{E}}{1-\hat{\alpha}^2} + \sqrt{\frac{\hat{E}^2}{(1-\hat{\alpha}^2)^2} - \frac{\hat{E}_0^2}{1-\hat{\alpha}^2}} \cos\sqrt{1-\hat{\alpha}^2}\theta \\ &= \frac{\alpha E}{\ell^2 c^2 - \alpha^2} + \sqrt{\frac{E^2 c^2 \ell^2}{(c^2\ell^2-\alpha^2)^2} - \frac{m_0^2 c^4}{c^2\ell^2-\alpha^2}} \cos\sqrt{1-\frac{\alpha^2}{c^2\ell^2}}\theta. \end{aligned} \tag{4.40}$$

It is easy to check that in the nonrelativistic limit, $c^2 \to \infty$, $E \to m_0 c^2 + \tilde{E}$, the previous equation (2.35) is recovered. The condition for a bound orbit, so that r is always finite, is that

$$\hat{E}^2 - \hat{E}_0^2(1-\hat{\alpha}^2) < \hat{\alpha}^2 \hat{E}^2, \tag{4.41}$$

or

$$\frac{\hat{E}}{\hat{E}_0} < 1. \tag{4.42}$$

If this is not satified, there are lines along which the particle enters and leaves the system, in other words, scattering states. Some examples are provided in the figures.

Further discussion of the relativistic Kepler/Coulomb problem can be found in T. Boyer, Am. J. Phys. **72**, 992 (2004). Of course, a correct treatment of relativistic planetary orbits requires general relativistic concepts.

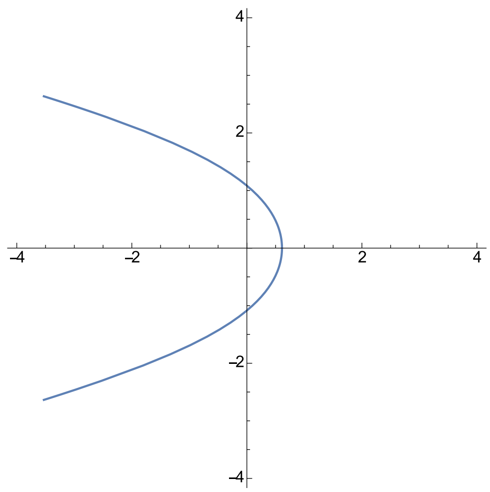

Figure 4.2: Unbound relativistic orbit in a $-\alpha/r$ potential. The parameters are $\hat{\alpha} = 0.5$, $\hat{E} = 1.1$, $\hat{E}_0 = 1$.

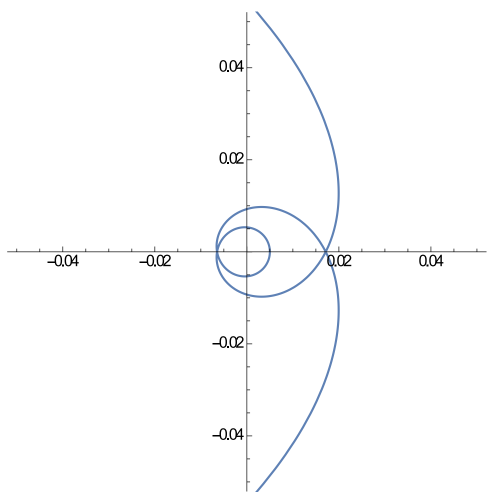

Figure 4.3: Unbound relativistic orbit in a $-\alpha/r$ potential. The parameters are $\hat{\alpha} = 0.95$, $\hat{E} = 10$, $\hat{E}_0 = 1$. Note that the orbit is open, but encircles the origin twice.

4.3 Problems for Chapter 4

1. Show that the form of the Hamiltonian in terms of p_r and p_θ contained in Eq. (4.33) follows from the general canonical formulation given in Sec. 2.2. That is, show that

$$H = c\sqrt{p_r^2 + \frac{p_\theta^2}{r^2} + m_0^2 c^2} - \frac{\alpha}{r}, \tag{4.43}$$

   starting from the Lagrangian (4.28).

2. Check that the formula (4.40) reduces to the nonrelativistic limit (2.35) when $c \to \infty$.

3. Show there is a lower bound on ℓ in order that a circular orbit can exist in the relativistic Coulomb problem.

4. Work out the relativistic orbits for the Coulomb problem in the cases $\ell = \alpha/c$ and $\ell < \alpha/c$. In particular, what happens when $\ell = 0$?