Chapter 3

Relativistic Kinematics

Recall that we briefly discussed Galilean boosts, transformation going from one inertial frame to another one, the first moving with an infinitesimal velocity $\delta\mathbf{v}$ with respect to the second:

$$\delta\mathbf{r}_a = \delta\mathbf{v}t, \quad \delta\mathbf{p}_a = m_a\delta\mathbf{v}, \quad \delta t = 0. \tag{3.1}$$

The generator of the boost was

$$\mathbf{N} = \mathbf{P}t - M\mathbf{R}, \tag{3.2}$$

where $\mathbf{P}$ is the total momentum, $\mathbf{R}$ is the position vector of the center of mass,

$$\mathbf{R} = \frac{1}{M}\sum_a m_a\mathbf{r}_a, \quad M = \sum_a m_a. \tag{3.3}$$

The generator being conserved, $d\mathbf{N}/dt = 0$, although not as a result of an invariance of the action, implies that

$$\mathbf{P} = M\mathbf{V}, \quad \mathbf{V} = \frac{d\mathbf{R}}{dt}. \tag{3.4}$$

However, Einstein gave up the inviolability of time, adhered to by Galileo and Newton. In fact, an *event* is specified by a point in spacetime, $(t, \mathbf{x})$; since we conventionally use different units for time and distance, we introduce a constant c (the speed of light), so that the four-dimensional coordinate of a point in spacetime is

$$x^\mu = (ct, \mathbf{x}), \quad x^0 = ct, \quad x^1 = x, \quad x^2 = y, \quad x^3 = z. \tag{3.5}$$

Under a rotation, the length of a (three-) vector is unchanged,

$$x^2 = \mathbf{x}\cdot\mathbf{x}, \quad \delta(\mathbf{x}\cdot\mathbf{x}) = 2(\delta\boldsymbol{\omega}\times\mathbf{x})\cdot\mathbf{x} = 0. \tag{3.6}$$

Similarly, under a boost, the 4-dimensional "length" is unchanged:

$$s^2 = \sum_{\mu,\nu=0}^{3} x^\mu x^\nu g_{\mu\nu} \equiv x^\mu x^\nu g_{\mu\nu} = x^\mu x_\nu = \mathbf{x}\cdot\mathbf{x} - c^2 t^2 \tag{3.7}$$

(this is called the "proper" distance between the origin and the spacetime point $(ct, \mathbf{x})$), where we have introduced the Einstein summation convention of summing over repeated upper (contravariant) and lower (covariant) indices, and the metric tensor in Cartesian coordinates:

$$g_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1), \tag{3.8}$$

which lowers indices,

$$x_\mu = g_{\mu\nu} x^\nu = (-ct, \mathbf{x}). \tag{3.9}$$

If we demand that s^2 be invariant under an infinitesimal linear transformation,

$$\delta x^\mu = \delta\omega^{\mu\nu} x_\nu, \quad \delta\omega^{\mu\nu} = \text{constant}, \tag{3.10}$$

that is,

$$\delta s^2 = \delta x^\mu x_\mu + x^\mu \delta x_\mu = 2\delta x^\mu x_\mu = 2\delta\omega^{\mu\nu} x_\nu x_\mu = 0, \tag{3.11}$$

this implies

$$\delta\omega^{\mu\nu} = -\delta\omega^{\nu\mu}, \tag{3.12}$$

that is, $\delta\omega^{\mu\nu}$ is *antisymmetric*. This means there are only 6 independent infinitesimal parameters,

$$\delta\omega^{12}, \quad \delta\omega^{23}, \quad \delta\omega^{31}, \quad \delta\omega^{01}, \quad \delta\omega^{02}, \quad \delta\omega^{03}. \tag{3.13}$$

The first three correspond to rotations we discussed before, for example,

$$\delta\omega_{12} = \delta\omega_3 \tag{3.14}$$

corresponds to a rotation in the 12 plane, or a rotation about the 3 axis, or in general[1]

$$\delta\omega_{ij} = \epsilon_{ijk}\delta\omega_k, \tag{3.15}$$

is a rotation in the ij plane, or about the k axis. Here we introduced the totally antisymmetric Levi-Civita symbol

$$\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij} = -\epsilon_{jik} = -\epsilon_{ikj} = -\epsilon_{kji}, \tag{3.16}$$

with

$$\epsilon_{123} = 1. \tag{3.17}$$

What are new are the $\delta\omega^{0i}$; these are the Einsteinian boosts or Lorentz transformations. So, for a boost in the z direction,

$$\delta x^0 = \delta\omega^{03} z, \quad \delta z = \delta\omega^{30} x_0, \tag{3.18}$$

[1] Note that Greek indices run from 0 to 3, while Latin indices run from 1 to 3.

or letting $\delta\omega^{03} = \frac{1}{c}\delta v$,

$$\delta z = \delta v t, \quad \delta t = \frac{1}{c^2}\delta v z, \quad \delta x = \delta y = 0. \tag{3.19}$$

Now, unlike with the Galilean boost (3.1), time changes, but the effect is very small for low velocities, $\delta v/c \ll 1$. This looks like a sort of infinitesimal rotation, through an imaginary angle. Let $\delta\theta = \delta v/c$. Then we can write the above transformation as a set of differential equations,

$$\frac{dz}{d\theta} = ct, \quad \frac{dct}{d\theta} = z, \tag{3.20}$$

which can be combined into the second-order equations,

$$\frac{d^2}{d\theta^2}z = z, \quad \frac{d^2}{d\theta^2}ct = ct. \tag{3.21}$$

If $z(0)$ and $ct(0)$ denote the values of z and ct at $\theta = 0$, the solution of these equations is

$$z(\theta) = z(0)\cosh\theta + ct(0)\sinh\theta, \quad ct(\theta) = ct(0)\cosh\theta + z(0)\sinh\theta. \tag{3.22}$$

This represents a general boost or Lorentz transformation along the z direction. What is the significance of θ? Because $|v/c| < 1$, the relation between the relative velocity of the two frames and the parameter θ must be

$$\tanh\theta = \frac{v}{c}, \tag{3.23}$$

which lies between -1 and $+1$ for all real θ. Because of the identity

$$\cosh^2\theta - \sinh^2\theta = 1, \tag{3.24}$$

we see that

$$\cosh\theta = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}, \quad \sinh\theta = \frac{v}{c}\left(1 - \frac{v^2}{c^2}\right)^{-1/2}. \tag{3.25}$$

Let us introduce the abbreviations

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad \beta = \frac{v}{c}. \tag{3.26}$$

Then the Lorentz transformation along the z axis reads

$$z' = \gamma(z + \beta ct), \quad t' = \gamma(t + \beta z/c). \tag{3.27}$$

This describes a boost from one frame to another moving relevant to the first with a velocity $-v$ along the z axis.

To verify that $\mathbf{x}^2 - c^2t^2$ is indeed invariant under a finite boost, we can write the above transformation in the following form,

$$z' + ct' = \gamma[z(1+\beta) + ct(1+\beta)] = \sqrt{\frac{1+\beta}{1-\beta}}(z+ct), \quad z' - ct' = \sqrt{\frac{1-\beta}{1+\beta}}(z-ct), \tag{3.28}$$

from which the claimed invariance is immediate.

What about two succesive boosts in the same direction? First,

$$z' + ct' = \sqrt{\frac{1+\beta_1}{1-\beta_1}}(z+ct), \tag{3.29}$$

and then

$$z'' + ct'' = \sqrt{\frac{1+\beta_2}{1-\beta_2}}(z'+ct') = \sqrt{\frac{1+\beta_2}{1-\beta_2}}\sqrt{\frac{1+\beta_1}{1-\beta_1}}(z+ct), \tag{3.30}$$

but the result must be a single boost with a velocity $v = \beta c$,

$$z'' + ct'' = \sqrt{\frac{1+\beta}{1-\beta}}(z+ct). \tag{3.31}$$

Thus

$$\frac{1+\beta}{1-\beta} = \frac{1+\beta_1}{1-\beta_1}\frac{1+\beta_2}{1-\beta_2} = \frac{1+\beta_1+\beta_2+\beta_1\beta_2}{1-\beta_1-\beta_2+\beta_1\beta_2} = \frac{1+\frac{\beta_1+\beta_2}{1+\beta_1\beta_2}}{1-\frac{\beta_1+\beta_2}{1+\beta_1\beta_2}}, \tag{3.32}$$

so

$$\beta = \frac{\beta_1+\beta_2}{1+\beta_1\beta_2}. \tag{3.33}$$

This gives the relativistic law for the addition of velocities in the same direction:

$$v = \frac{v_1+v_2}{1+v_1v_2/c^2}, \tag{3.34}$$

which is usual addition if $|v_a/c| \ll 1$, but has the property that if one of the velocities is that of light, say $v_2 = c$, then $v = c$, and the sum of two subluminal velocities is always less than that of light in magnitude.

3.1 Generators

Let us consider a single free particle, and generalize the infinitesimal transformations to a boost $\delta\mathbf{v}$. The transformations are, instead of Eq. (3.1),

$$\delta\mathbf{r} = \delta\mathbf{v}t, \quad \delta\mathbf{p} = m\delta\mathbf{v}, \quad \delta t = \frac{1}{c^2}\delta\mathbf{v}\cdot\mathbf{r}. \tag{3.35}$$

The integrand in the action is

$$dt\,L = \mathbf{p}\cdot d\mathbf{r} - H\,dt, \tag{3.36}$$

which changes by ($H = H(\mathbf{p})$)

$$\begin{aligned}\delta[dt\,L] &= m\delta\mathbf{v}\cdot d\mathbf{r} + \mathbf{p}\cdot\delta\mathbf{v}dt - \frac{\partial H}{\partial \mathbf{p}} m\delta\mathbf{v}dt - H\frac{\delta\mathbf{v}}{c^2}\cdot d\mathbf{r}\\ &= \delta\mathbf{v}\cdot\left[\left(m - \frac{H}{c^2}\right)d\mathbf{r} + \left(\mathbf{p} - m\frac{\partial H}{\partial\mathbf{p}}\right)dt\right].\end{aligned} \tag{3.37}$$

This will be zero, so the action is *invariant*, if

$$m = \frac{H}{c^2} \quad (\text{or} \quad E = mc^2) \quad \text{and} \quad \frac{\partial H}{\partial\mathbf{p}} = \frac{\mathbf{p}}{m} = \frac{\mathbf{p}}{H}c^2, \tag{3.38}$$

or

$$H\,dH = c^2\mathbf{p}\cdot d\mathbf{p}, \tag{3.39}$$

which means

$$H^2 = p^2c^2 + \text{constant} = p^2c^2 + m_0^2c^2, \tag{3.40}$$

where we have introduced the rest mass, m_0. The mass m which appears in the momentum is not the rest mass, but c^2 divided into

$$H = \sqrt{p^2c^2 + m_0^2c^4}. \tag{3.41}$$

The generator of a boost or Lorentz transformation is

$$G = \mathbf{p}\cdot\delta\mathbf{r} - H\delta t = \mathbf{p}\cdot\delta\mathbf{v}t - H\frac{\delta\mathbf{v}\cdot\mathbf{r}}{c^2} = \delta\mathbf{v}\cdot\mathbf{N}, \tag{3.42}$$

where

$$\mathbf{N} = \mathbf{p}t - \frac{H}{c^2}\mathbf{r}. \tag{3.43}$$

The constancy of G implies that of $\mathbf{N}$, where, because $\mathbf{p}$ and H are constants, implies

$$0 = \frac{dN}{dt} = \mathbf{p} - \frac{H}{c^2}\frac{d\mathbf{r}}{dt}, \tag{3.44}$$

or, since $H = mc^2$,

$$\mathbf{p} = m\mathbf{v}. \tag{3.45}$$

In terms of velocity, this means

$$p^2 = (p^2 + m_0^2c^2)\frac{v^2}{c^2}, \tag{3.46}$$

or

$$p^2\left(1 - \frac{v^2}{c^2}\right) = m_0^2v^2, \tag{3.47}$$

that is,

$$\mathbf{p} = \frac{m_0\mathbf{v}}{\sqrt{1 - v^2/c^2}} = \gamma m_0\mathbf{v}. \tag{3.48}$$

In other words, $m = m_0\gamma$, or

$$H = mc^2 = m_0\gamma c^2. \tag{3.49}$$

3.2 Action

Now the Lagrangian is

$$L = \mathbf{p}\cdot\dot{\mathbf{r}} - H = m_0\gamma v^2 - m_0\gamma c^2 = -m_0c^2\sqrt{1-\frac{v^2}{c^2}}, \tag{3.50}$$

as we saw before.

Now recall we define a proper distance as

$$s^2 = \mathbf{x}^2 - c^2t^2, \tag{3.51}$$

so for an infinitesimal interval, can define a proper distance, or, alternatively, a proper time,

$$c^2d\tau^2 = -ds^2 = c^2dt^2 - d\mathbf{x}^2, \tag{3.52}$$

or, in terms of $d\mathbf{x}/dt = \mathbf{v}$,

$$d\tau = dt\sqrt{1-\frac{v^2}{c^2}}. \tag{3.53}$$

Thus the action in proportional to the proper time along the path of the particle,

$$W_{12} = -m_0c^2\int_{t_2}^{t_1} dt\sqrt{1-\frac{v^2}{c^2}} = -m_0c^2\int_2^1 d\tau = -m_0c\int_2^1\sqrt{-dx^\mu dx_\mu}. \tag{3.54}$$

Now let's use the action principle: since the action does not depend upon τ, except implicitly through $x^\mu(\tau)$, the variation of the action is merely

$$\begin{aligned}\delta W_{12} &= -m_0c\int_2^1\left(-\frac{1}{2}\right)(-dx_\mu dx^\mu)^{-1/2}2d\delta x^\mu dx_\mu \\ &= m_0\int_2^1 d\delta x^\mu\frac{dx_\mu}{d\tau} \\ &= m_0\int_2^1\left[d\left(\delta x^\mu\frac{dx_\mu}{d\tau}\right) - \delta x^\mu\frac{d^2}{d\tau^2}x_\mu d\tau\right],\end{aligned} \tag{3.55}$$

from which we infer

$$G = m_0\frac{dx^\mu}{d\tau}\delta x_\mu \equiv p^\mu\delta x_\mu, \tag{3.56}$$

and the equation of motion for a free, relativistic particle,

$$\frac{d^2x^\mu}{d\tau^2} = 0. \tag{3.57}$$

Here appears the four-momentum,

$$p^\mu = m_0\frac{dx^\mu}{d\tau} = (E/c, \mathbf{p}) = \gamma m_0(c, \mathbf{v}). \tag{3.58}$$

The square of the four-momentum is an invariant scalar,

$$p^\mu p_\mu = -m_0^2c^2, \tag{3.59}$$

the negative square of the rest energy m_0c^2, divided by c^2. The equation of motion here is just the conservation of energy and momentum of a free particle. And it is easy to check that the momentum is indeed the canonical one:

$$p^\mu = \frac{\partial L}{\partial \dot{x}_\mu}, \quad \dot{x}_\mu \equiv \frac{d}{d\tau} x_\mu, \tag{3.60}$$

because

$$\frac{\partial}{\partial \dot{x}_\mu}\left(-m_0 c\sqrt{-\dot{x}^\mu \dot{x}_\mu}\right) = \frac{m_0 c}{\sqrt{-\dot{x}^\nu \dot{x}_\nu}}\frac{dx^\mu}{d\tau} = m_0 \frac{dx^\mu}{d\tau}, \tag{3.61}$$

where we've defined the Lagrangian by

$$W_{12} = -m_0 c^2 \int_2^1 d\tau \sqrt{-d\dot{x}_\mu d\dot{x}^\mu} = \int_2^1 d\tau\, L; \tag{3.62}$$

L is $-m_0c^2$ on the true trajectory, but is a function of the four-velocity elsewhere.

3.3 Problems for Chapter 3

1. Find the explicit form of θ given in Eq. (3.23) in terms of v/c as a logarithm. How does the composition law of velocities appear in terms of θ?

2. Show that the finite Lorentz transformations have the vectorial form

$$\mathbf{r}' = \mathbf{r} + \frac{\gamma^2}{1+\gamma}\frac{1}{c^2}\mathbf{v}\mathbf{v}\cdot\mathbf{r} + \gamma\mathbf{v}t, \tag{3.63a}$$

$$t' = \gamma\left(t + \frac{1}{c^2}\mathbf{v}\cdot\mathbf{r}\right). \tag{3.63b}$$

   Check the invariance of $\mathbf{r}^2 - (ct)^2$.

3. A body of mass M is at rest relative to one observer. Two photons, each of energy ϵ, moving in opposite directions along the x-axis, fall on the body and are absorbed. Since the photons carry equal and opposite momenta, no net momentum is transferred to the body, and it remains at rest. Another observer is moving relative to the first slowly along the y-axis. Relative to her, the two photons and the body, both before and after the absorption act, have a common velocity $\mathbf{v}$ ($|\mathbf{v}| \ll c$) along the y-axis. Reconcile conservation of the y-component of momentum with the fact that the velocity of the body does not change when the photons are absorbed.