Chapter 13

Translations and Rotations

We now make explicit reference to the arena of space and time. A coordinate system is an idealization of an experimental apparatus. We isolate what is significant for the physical system, not just a specific property of the apparatus.

There are a number of transformations we can make on a coordinate system

- We can translate the coordinate system (spatial displacement).
- We can rotate the coordinate system.
- We can go to a coordinate moving with constant relative velocity (a "boost" or Galilean transformation).
- We can displace the origin of time.

Under translations, rotations, etc., the physics is not altered. We have a great freedom in choosing our coordinate system—different coordinate systems give different, but equivalent, descriptions.

We saw, in quantum mechanics, the freedom to make unitary transformations, which preserve algebraic properties and numbers. Intrinsic relations are independent of the coordinate system; in quantum mechanics, unitary transformations preserve intrinsic relations.

What's a unitary operator got to do with a physical property? Recall that under a unitary transformation, states and operators change like this:

$$\bar{X} = U^{-1} X U, \quad \overline{\langle\,|} = \langle\,| U, \quad \overline{|\,\rangle} = U^{-1} |\,\rangle. \tag{13.1}$$

A physical property is represented by a *Hermitian* operator, $H = H^\dagger$. Recall that a unitary operator can be written as

$$U = e^{iH}, \quad U^\dagger = e^{-iH} = U^{-1}, \quad U^\dagger U = 1. \tag{13.2}$$

Any unitary operator can be written this way. We'll see this, by considering *small changes*. For example, differential equations are simple, whereas finite changes can be complicated.

Under a small change (of a coordinate system, say), the unitary operator differs just a little bit from the unit operator. For no change, $U = 1$. For a small change

$$U = 1 + iG, \tag{13.3}$$

where G, called the generator, contains a small parameter. The i is present to make the result simpler. What restrictions are there on G so that U is unitary? Given the above form for U, its adjoint is

$$U^\dagger = 1 - iG^\dagger, \tag{13.4}$$

and when we insist

$$1 = U^\dagger U = (1 - iG^\dagger)(1 + iG) = 1 + i(G - G^\dagger) + G^\dagger G, \tag{13.5}$$

where the last quantity is second order in the implicity small parameter, so negligible. Thus we conclude

$$G = G^\dagger, \tag{13.6}$$

or G is Hermitian. Indeed, if we start from

$$e^{iH} = 1 + iH + \frac{(iH)^2}{3} + \ldots \approx 1 + iH, \tag{13.7}$$

if H is small.

We will now get physical properties from coordinate system changes:

- Displacements correspond to momentum.
- Rotations correspond to angular momentum.
- Time displacements correspond to energy.

For these fundamental physical properties, we will borrow *only* names from classical physics.

How do we build up unitary transformations? If U_1, U_2 are unitary, so is U_1U_2:

$$(U_1U_2)^\dagger U_1U_2 = U_2^\dagger U_1^\dagger U_1U_2 = U_2^\dagger U_2 = 1. \tag{13.8}$$

For infinitesimal transformations,

$$U_1U_2 = (1 + iG_2)(1 + iG_2) = 1 + i(G_2 + G_2), \tag{13.9}$$

if $-G_1G_2$ is neglected, as small. The product of two infinitesimal transformations is an infinitesimal transformation. The generator of the product is the sum of the generators.

What about dimensions? G is obviously dimensionless. However, when G is identified in terms of physical properties,

$$G(\text{atomic units}) = \frac{1}{\hbar} G(\text{conventional units}), \tag{13.10}$$

where $\hbar = h/(2\pi)$, the "quantum of action," having units of momentum times distance or energy times time, is a conversion factor to convert from conventional units to atomic units. We will use both sets of units.

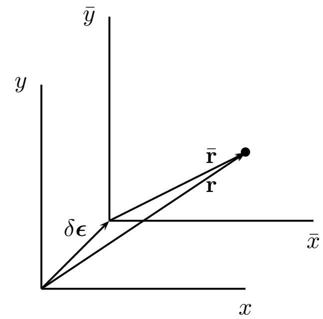

Figure 13.1: A point in space as described in two different coordinate systems, O and $\bar{O}$, where the latter's origin is displaced from the former's by an amount $\delta\boldsymbol{\epsilon}$.

13.1 Spatial translations

Consider a displacement of the origin of the coordinate system, first by an amount $\delta\epsilon$ along the x axis. A point in the two coordinate system is represented by (x, y), and $(\bar{x}, \bar{y})$, respectively, where

$$\bar{x} = x - \delta\epsilon, \quad \bar{y} = y. \tag{13.11}$$

More generally, if the displacement of the origin is by an infinitesimal vector $\delta\boldsymbol{\epsilon} = (\delta\epsilon_x, \delta\epsilon_y, \delta\epsilon_z)$, as shown in Fig. 13.1

$$\bar{x} = x - \delta\epsilon_x, \quad \bar{y} = y - \delta\epsilon_y, \quad \bar{z} = z - \delta\epsilon_z, \tag{13.12a}$$

$$\text{or } \bar{\mathbf{r}} = \mathbf{r} - \delta\boldsymbol{\epsilon}, \tag{13.12b}$$

expressing the new position vector in terms of the old position vector, and the vector displacement of the origin. If we have a transformation which is parametrized by $\delta\epsilon_x$, the corresponding unitary operator is

$$U = 1 + iG, \quad G_{\delta\epsilon_x} = \delta\epsilon_x P_x. \tag{13.13}$$

We call P_x is the x-component of linear momentum. Likewise, if the origin was displaced along y,

$$G_{\delta\epsilon_y} = \delta\epsilon_y P_y, \tag{13.14}$$

or along the z axis,

$$G_{\delta\epsilon_z} = \delta\epsilon_z P_z, \tag{13.15}$$

So for a general infinitesimal displacement of the origin,

$$G_{\delta\boldsymbol{\epsilon}} = \delta\epsilon_x P_x + \delta\epsilon_y P_y + \delta\epsilon_z P_z = \delta\boldsymbol{\epsilon} \cdot \mathbf{P}. \tag{13.16}$$

Since the displacement $\delta\boldsymbol{\epsilon}$ is real, the momentum $\mathbf{P}$ is Hermitian.

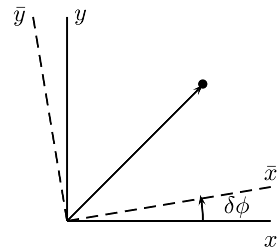

Figure 13.2: A point in space as described in two different coordinate systems, O and $\bar{O}$, where the two coordinate systems differ by a rotation through an angle $\delta\phi$ about the z axis.

13.2 Rotations

We will have much more to say about translations, but let us turn to rotations. Figure 13.2 shows a rotation of the coordinate system in the x-y plane. New coordinates are obtained from old coordinates as follows:

$$\bar{x} = x\cos\phi + y\sin\phi, \quad \bar{y} = -x\sin\phi + y\cos\phi, \tag{13.17}$$

which preserves the length of the position vector:

$$\bar{x}^2 + \bar{y}^2 = x^2 + y^2. \tag{13.18}$$

Let's now consider an infinitesimal rotation,

$$\phi \to \delta\phi, \quad \sin\phi \to \delta\phi, \quad \cos\phi \to 1. \tag{13.19}$$

Then the above rotation becomes

$$\bar{x} = x + y\delta\phi, \quad \bar{y} = y - x\delta\phi, \quad \bar{z} = z. \tag{13.20}$$

You can see this directly without using sines and cosines.

The above is a rotation about the z axis. Introduce the infinitesimal vector

$$\delta\boldsymbol{\omega} = (0, 0, \delta\phi), \tag{13.21}$$

where $\delta\boldsymbol{\omega}$ specifies both the direction of the rotation, and the magnitude of the rotation. Note that

$$\delta\boldsymbol{\omega} \times \mathbf{r} = (-\delta\phi\, y, \delta\phi\, x, 0). \tag{13.22}$$

So the above is unified as

$$\bar{\mathbf{r}} = \mathbf{r} - \delta\boldsymbol{\omega} \times \mathbf{r}. \tag{13.23}$$

This is valid no matter what direction the axis of rotation has.

Unitary transformations are made up of corresponding infinitesimal generators, that is, we add up the z, x, and y components of the rotations:

$$G_{\delta\boldsymbol{\omega}} = \delta\omega_x J_x + \delta\omega_y J_y + \delta\omega_z J_z = \delta\boldsymbol{\omega} \cdot \mathbf{J}. \tag{13.24}$$

J is a Hermitian operator having the dimensions of action. Existence of these operators is dictated by quantum mechanics; classical mechanics supplies the name only: **J** is called the angular momentum operator.

Rotations are simpler than translations because we can make more general statements about them.

Any vector transforms under a rotation like the coordinate (position) vector. How do operators, like **P**, change under an infinitesimal transformation? Recall

$$\bar{X} = U^{-1} X U \tag{13.25}$$

preserves all algebraic relations, if we transform all operators the same way:

$$\begin{aligned} X + Y &= Y \to \bar{X} + \bar{Y} = \bar{Z}, && (13.26a)\\ XY &= Z \to \bar{X}\bar{Y} = \bar{Z}, && (13.26b) \end{aligned}$$

etc. The two descriptions, barred and unbarred, are equivalent. For infinitesimal transformations, in conventional units,

$$U = 1 + \frac{i}{\hbar} G, \quad U^\dagger = U^{-1} = 1 - \frac{i}{\hbar} G, \tag{13.27}$$

so

$$\bar{X} = \left(1 - \frac{i}{\hbar} G\right) X \left(1 + \frac{i}{\hbar} G\right) = X + \frac{i}{\hbar}[X, G], \tag{13.28}$$

neglecting second-order infinitesimals. On the other hand, X only changes a little bit,

$$\bar{X} = X - \delta X; \tag{13.29}$$

for example, under a displacement, we wrote $\bar{\mathbf{r}} = \mathbf{r} - \delta\boldsymbol{\epsilon}$. So we conclude

$$\delta X = \frac{1}{i\hbar}[X, G]. \tag{13.30}$$

If we know what δX is (as we do for vectors), we learn about commutators, and as we'll see, compatibility of physical properties.

If X is Hermitian, so must be $\bar{X}$. Consider $\frac{1}{i}[X, Y]$; this is Hermitian if X and Y are.:

$$\begin{aligned} \left(\frac{1}{i}[X, Y]\right)^\dagger &= \left(\frac{1}{i}(XY - YX)\right)^\dagger = -\frac{1}{i}(Y^\dagger X^\dagger - X^\dagger Y^\dagger)\\ &= \frac{1}{i}[X^\dagger, Y^\dagger] = \frac{1}{i}[X, Y], && (13.31) \end{aligned}$$

if $X^\dagger = X$, $Y^\dagger = Y$. Indeed, δX is Hermitian because X and G are both Hermitian.

Now consider a scalar S and a vector $\mathbf{V}$. How do these respond under a rotation of the coordinate system?

$$\begin{aligned} \bar{S} &= S, && (13.32a)\\ \bar{\mathbf{V}} &= \mathbf{V} - \delta\boldsymbol{\omega} \times \mathbf{V}, && (13.32b) \end{aligned}$$

the latter modeled on the response (13.23) of the position vector. If these are expressed in the general form

$$\bar{X} = X - \delta X, \tag{13.33}$$

we have

$$\delta S = 0, \quad \delta \mathbf{V} = \delta \boldsymbol{\omega} \times \mathbf{V}. \tag{13.34}$$

If these are operators, or physical quantities,

$$\delta S = \frac{1}{i\hbar}[S, \delta\boldsymbol{\omega} \cdot \mathbf{J}] = 0, \tag{13.35a}$$

$$\delta \mathbf{V} = \frac{1}{i\hbar}[\mathbf{V}, \delta\boldsymbol{\omega} \cdot \mathbf{J}] = \delta\boldsymbol{\omega} \times \mathbf{V}. \tag{13.35b}$$

We therefore learn what commutators are. Since

$$\delta\boldsymbol{\omega} \cdot \mathbf{J} = \delta\omega_x J_x + \delta\omega_y J_y + \delta\omega_z J_z, \tag{13.36}$$

where the components $\delta\omega_x$, $\delta\omega_y$, $\delta\omega_z$ are arbitrary, we have for a scalar S

$$[S, J_x] = 0, \quad [S, J_y] = 0, \quad [S, J_z] = 0, \tag{13.37}$$

or

$$[S, \mathbf{J}] = 0. \tag{13.38}$$

For example, $\mathbf{J}^2$ is a scalar, so we conclude

$$[\mathbf{J}^2, \mathbf{J}] = 0. \tag{13.39}$$

We have taken this for granted. For example, this says

$$[\mathbf{J}^2, J_z] = 0, \tag{13.40}$$

which says that $\mathbf{J}^2$ and J_z are compatible physical properties. That is, we can measure the magnitude, and the z component of $\mathbf{J}$ simultaneously. So we can talk about $J = 1/2$, a spin-1/2 atom, and $J_z = 1/2$, simultaneously.

For a vector, from Eq. (13.35b),

$$\frac{1}{i\hbar}[\mathbf{V}, \delta\boldsymbol{\omega} \cdot \mathbf{J}] = \delta\boldsymbol{\omega} \times \mathbf{V}. \tag{13.41}$$

Suppose we consider a rotation about the z axis, $\delta\boldsymbol{\omega} = \hat{\mathbf{z}}\delta\omega_z$. Then the x-component of Eq. (13.41) is

$$\frac{1}{i\hbar}[V_x, J_z]\delta\omega_z = -\delta\omega_z V_y, \tag{13.42a}$$

and the y component is

$$\frac{1}{i\hbar}[V_y, J_z]\delta\omega_z = \delta\omega_z V_x, \tag{13.42b}$$

while the z component vanishes,

$$\frac{1}{i\hbar}[V_z, J_z]\delta\omega_z = 0. \tag{13.42c}$$

This last says that a rotation about the z axis does not change the z component, which just repeats what we did before for $\bar{x}$ and $\bar{y}$. Thus we have the following commutation relations for the components of a vector:

$$[V_z, J_z] = 0, \tag{13.43a}$$

$$\frac{1}{i\hbar}[V_y, J_z] = V_x, \tag{13.43b}$$

$$\frac{1}{i\hbar}[V_x, J_z] = -V_y, \tag{13.43c}$$

which says that parallel components commute, and perpendicular components have a positive sign when the components are in cyclic order. Thus, we anticipate

$$\frac{1}{i\hbar}[V_x, J_y] = V_z, \tag{13.44}$$

which is obtained directly by considering a rotation about the y axis.

As an application of this general rule for vectors, consider $\mathbf{V} = \mathbf{J}$. Then we must have

$$\frac{1}{i\hbar}[J_x, J_y] = J_z, \tag{13.45}$$

and so on by cyclic permutations. This can either be thought of as a rotation about the y axis, which mixes x and z components together, or as a rotation about the x axis, which mixes y and z components together. The cyclic permutations of this statement are

$$\frac{1}{i\hbar}[J_y, J_z] = J_x, \quad \frac{1}{i\hbar}[J_z, J_x] = J_y. \tag{13.46}$$

There is nothing sacred about the axes x, y, z; they are just a set of perpendicular directions, with a certain (right-handed) sense. In these formulas, $\mathbf{J}$ has two roles, 1) as a generator, and 2) as a vector. Another way of writing this result is obtained by noting

$$J_x J_y - J_y J_x = (\mathbf{J} \times \mathbf{J})_z, \tag{13.47}$$

so we can write the three angular momentum commutation relations as the single vector relation

$$\frac{1}{i\hbar}\mathbf{J} \times \mathbf{J} = \mathbf{J}. \tag{13.48}$$

We met this long ago, for spin-1/2, where

$$\mathbf{J} = \frac{\hbar}{2}\boldsymbol{\sigma}, \tag{13.49}$$

so Eq. (13.48) becomes

$$\frac{1}{2}\boldsymbol{\sigma}\times\frac{1}{2}\boldsymbol{\sigma}=i\frac{1}{2}\boldsymbol{\sigma}. \tag{13.50}$$

We had also seen the same for spin-1.

Although $\mathbf{J}^2$ and J_z commute and are therefore compatible, is J_x also compatible with these two? No, because

$$[J_x, J_z] \neq 0. \tag{13.51}$$

What can we say about the states in which $\mathbf{J}^2$, J_z have definite values? These are simultaneous eigenstates of these two operators, call them $|jm\rangle$, which satisfy the eigenvalue equations

$$\mathbf{J}^2|jm\rangle = \mathbf{J}^{2\prime}|jm\rangle, \tag{13.52a}$$

$$J_z|jm\rangle = J_z'|jm\rangle, \tag{13.52b}$$

where let us write

$$\mathbf{J}^{2\prime}=\hbar^2 j(j+1), \quad J_z'=\hbar m, \tag{13.53}$$

where at this point m and $j(j+1)$ are unknown numbers. (We can always choose $j \geq 0$.) The units of angular momentum, those of action, are taken care of by the appearance of $\hbar$. Now use the operator properties

$$\frac{1}{i\hbar}[J_x, J_z] = -J_y, \tag{13.54a}$$

$$\frac{1}{i\hbar}[J_y, J_z] = J_x. \tag{13.54b}$$

This represents a rotation in the x-y plane. It is useful to introduce non-Hermitian quantities

$$J_x + iJ_y = J_+, \quad J_x - iJ_y = J_-, \tag{13.55}$$

which are adjoints of each other,

$$J_+^\dagger = J_-, \quad J_-^\dagger = J_+. \tag{13.56}$$

Now

$$\frac{1}{i\hbar}[J_\pm, J_z] = -J_y \pm iJ_x = \pm i(J_x \pm iJ_y) = \pm iJ_\pm, \tag{13.57}$$

or

$$[J_+, J_z] = -\hbar J_+, \quad [J_-, J_z] = \hbar J_-, \tag{13.58}$$

which are adjoints of each other. When we write out the commutator explicitly, we have

$$J_zJ_+ = J_+(J_z+\hbar), \quad J_zJ_- = J_-(J_z-\hbar). \tag{13.59}$$

Take the first equation, and have it act on $|jm\rangle$,

$$J_zJ_+|jm\rangle = J_+(J_z+\hbar)|jm\rangle = \hbar(m+1)J_+|jm\rangle, \tag{13.60}$$

which says that

$$J_+|jm\rangle = C|j, m+1\rangle, \tag{13.61}$$

where the constant is present because $J_+|jm\rangle$ is not a unit vector. (Because of Eq. (13.39) J_+ does not change the value of $\mathbf{J}^2$.) This equation gives the changes in the values of J_z in units of $\hbar$. To determine C, multiply this equation by its adjoint

$$\langle jm|J_- = C^*\langle jm+1|, \tag{13.62}$$

so since $|jm+1\rangle$ is a unit vector,

$$\langle jm|J_-J_+|jm\rangle = |C|^2. \tag{13.63}$$

This equation says $|C|^2$ is a diagonal matrix element of J_-J_+, or the expectation value of J_-J_+. Now

$$J_\mp J_\pm = (J_x \mp iJ_y)(J_x \pm iJ_y) = J_x^2 + J_y^2 \pm i[J_x, J_y] = \mathbf{J}^2 - J_z^2 \mp \hbar J_z, \tag{13.64}$$

so we conclude

$$[J_+, J_-] = 2\hbar J_z, \tag{13.65}$$

which may directly confirmed:

$$[J_x + iJ_y, J_x - iJ_y] = i[J_y, J_x] - i[J_x, J_y] = 2\hbar J_z. \tag{13.66}$$

Thus, we see that J_-J_+ has a definite value in the state $|jm\rangle$,

$$J_-J_+|jm\rangle = \hbar^2[j(j+1) - m^2 - m]|jm\rangle. \tag{13.67}$$

In particular,

$$\langle jm|J_-J_+|jm\rangle = \hbar^2[j(j+1) - m(m+1)] = \hbar^2(j-m)(j+m+1), \tag{13.68}$$

or

$$|C|^2 = \hbar^2(j-m)(j+m+1). \tag{13.69}$$

This must be positive, so we must have $m \le j$. And, necessarily $j \ge 0$.

It is time to give these eigenvalues names: we call j the angular momentum quantum number, and m the magnetic quantum number, which makes reference to the Stern-Gerlach experiment. The above inequality roughly says that the component of a vector cannot exceed the length of the vector. Since J_+ has the effect of increasing the value of m by one unit, but $j-m$ cannot be negative, we conclude there must be a state with $m = m_{\text{max}}$, the maximum value of m:

$$J_+|jm_{\text{max}}\rangle = 0, \tag{13.70}$$

which says there is no state with $m = m_{\text{max}} + 1$. Now, the expression for $|C|^2$ tells us that

$$m_{\text{max}} = j. \tag{13.71}$$

Let us choose the phase so that

$$\frac{1}{\hbar}J_+|jm\rangle = \sqrt{(j-m)(j+m+1)}|jm+1\rangle. \tag{13.72}$$

We can also go down, using J_-:

$$J_zJ_- = J_-(J_z-\hbar), \quad \mathbf{J}^2J_- = J_-\mathbf{J}^2. \tag{13.73}$$

When these act on $|jm\rangle$,

$$J_z(J_-|jm\rangle) = (m-1)\hbar(J_-|jm\rangle), \tag{13.74a}$$
$$\mathbf{J}^2(J_-|jm\rangle) = j(j+1)\hbar^2(J_-|jm\rangle), \tag{13.74b}$$

so we conclude

$$J_-|jm\rangle = D|jm-1\rangle. \tag{13.75}$$

The adjoint of this equation is

$$\langle jm|J_+ = D^*\langle jm-1|. \tag{13.76}$$

Putting these two statements together, we see from Eq. (13.64),

$$\langle jm|J_+J_-|jm\rangle = |D|^2 = \hbar^2[j(j+1)-m^2+m] = \hbar^2(j+m)(j-m+1), \tag{13.77}$$

which is Eq. (13.69) with $m \to -m$. Since this cannot be negative, we conclude that $m \geq -j$, and that there must be a minimum value of m, $m_{\rm min}$ such that

$$J_-|jm_{\rm min}\rangle = 0, \tag{13.78}$$

and that $m_{\rm min} = -j$.

Choosing the phases in the simplest possible way,

$$\frac{1}{\hbar}J_-|jm\rangle = \sqrt{(j+m)(j-m+1)}|jm-1\rangle. \tag{13.79}$$

Equations (13.79), (13.72) together with

$$\frac{1}{\hbar}J_z|jm\rangle = m|jm\rangle, \tag{13.80a}$$
$$\frac{1}{\hbar^2}\mathbf{J}^2|jm\rangle = j(j+1)|jm\rangle, \tag{13.80b}$$

give the effect of J_x, J_y, J_z, and $\mathbf{J}^2$ on the state $|jm\rangle$.

For a given j, $m_{\rm min} = -j$, $m_{\rm max} = +j$. But $m_{\rm min}$ and $m_{\rm max}$ must differ by an integer, because repeated applications of J_+ must carry m from $m_{\rm min}$ to $m_{\rm max}$. Let the number of steps from $m_{\rm min}$ to $m_{\rm max}$ be n. Here are the first few examples:

- $n = 0$: $j = 0$, $m = 0$ (spin zero);

- $n = 1$: $j = -j + 1$ which implies $j = 1/2$, $m = 1/2$ or $-1/2$ (spin 1/2);
- $n = 2$: $j = -j + 2$, which implies $j = 1$, $m = 1, 0, -1$ (spin 1).

In general,

$$j = -j + n, \quad 2j = n, \quad j = \frac{n}{2}, \quad n = 0, 1, 2, 3, \ldots. \tag{13.81}$$

So the angular momentum quantum number takes the values 0, 1/2, 1, 3/2, 2, etc. The magnetic quantum number m tales on the $2j + 1$ values from $-j$ to j, through integer steps.

Let's finish this chapter by reminding you of spin 1/2. When $j = 1/2$, $m = \pm 1/2$, and we defined

$$\frac{1}{\hbar}\mathbf{J} = \frac{1}{2}\boldsymbol{\sigma}, \tag{13.82}$$

where the eigenvalues of σ_z are

$$\sigma_z' = \pm 1. \tag{13.83}$$

This property is true for the eigenvalues of σ_x, σ_y as well, so

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1. \tag{13.84}$$

which also follows from the eigenvalue

$$\frac{1}{\hbar}\mathbf{J}^{2\prime} = j(j+1) \Rightarrow \boldsymbol{\sigma}^2 = 3. \tag{13.85}$$

The J_+, J_- statements translate into

$$\frac{1}{2}(\sigma_x + i\sigma_y)|+\rangle = 0, \tag{13.86a}$$

$$\frac{1}{2}(\sigma_x + i\sigma_y)|-\rangle = |+\rangle, \tag{13.86b}$$

$$\frac{1}{2}(\sigma_x - i\sigma_y)|+\rangle = |-\rangle, \tag{13.86c}$$

$$\frac{1}{2}(\sigma_x - i\sigma_y)|-\rangle = 0. \tag{13.86d}$$

We express the coefficients appearing here as a matrix,

$$\langle\pm|\frac{1}{2}(\sigma_x + i\sigma_y)|\pm\rangle = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{13.87a}$$

$$\langle\pm|\frac{1}{2}(\sigma_x - i\sigma_y)|\pm\rangle = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{13.87b}$$

which are the adjoints (transposed, complex conjugates) of each other. If we add and subtract these, we get the familiar Pauli matrices,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{13.88}$$