Scaling Problems

Scaling problems are very popular in physics and so learning to solve these types of problems is important. An example of a scaling problem is:

The area of a circle is given by $A = \pi r^2$. If the radius(r) of a circle is increased by 20% what is the new area of the circle?

People may say that one cannot solve this problem since the original radius of the circle is unknown so it is impossible to determine the area of the circle. It is true that one cannot determine the original area of the circle, but one can determine how much larger the area of a circle will become after the radius is increased. This is done using a technique involving ratios. Let's solve the above problem using ratios.

We first write down what we know: We know the area of a circle is given by: $A = \pi r^2$. Now we need to write down in equation form what it means for the radius to increase by 20%. Often using specific examples can help in these types of problem, so let's write down some specific examples. If the original radius of a circle is 1.0 m, and the new radius is 20% larger, this means the new radius would be 1.2 m or (1.0 m+ 0.2*1.0 m). If the original radius of a circle is 2.0 m, the new radius would be 2.4 m or (2.0 m+0.2*2.0 m). We can start to see a pattern $r_{new} = r_{original} + 0.2 * r_{original} = 1.2 * r_{original}$. This pattern works for all changes in radius. For example, we can look at other examples: If the radius increased by 60% $r_{new} = 1.6 * r_{original}$, if the radius increased by 100% (i.e doubled) $r_{new} = 2.0 r_{original}$.

Now that we know how to write down in equation form the relationship between the original radius and the new radius we can go back to the original problem and solve it.

We know the relationship between the area and radius of a circle: $A_{original} = \pi r^2_{original}$ and $A_{new} = \pi r^2_{new}$. As mentioned above, we do not know the numerical value of the radius so we cannot actually compute the areas of the circles. However, by taking the ratio of the above equations we can solve the problem. Taking the ratio of the 2 equations above we see

$$\frac{A_{new}}{A_{original}} = \frac{\pi r^2_{new}}{\pi r^2_{original}}$$

.

From above we note that $r_{new} = 1.2r_{original}$.
So we can substitute this expression for r_{new} into our ratio

$$\frac{A_{new}}{A_{original}} = \frac{\pi(1.2r_{original})^2}{\pi r^2_{original}} = \frac{\pi(1.2)^2 r^2_{original}}{\pi r^2_{original}}.$$

Note that we can now cancel out the π and $r^2_{original}$ from the numerator and denominator leaving

$$\frac{A_{new}}{A_{original}} = (1.2)^2 = 1.44$$

or

$$A_{new} = 1.44A_{original}.$$

Or we see that the new area is 44% larger than the original area.

From this example, we can notice a few important features of taking ratios. Note that the value π cancelled out in the above ratio. So we note that any constants do not matter in the final answer. For example if we ask the question,

If $K = 0.5mv^2$ and v increases by 20%, how much larger is K. Note we don't actually need to know what K, m and v represent in this equation. All we need to know is the relationship and what variable is changing. Using the above technique we note $v_{new} = 1.2v_{original}$ so

$$\frac{K_{new}}{K_{original}} = \frac{0.5m(1.2v_{original})^2}{0.5m(v_{original})^2} = (1.2)^2 = 1.44$$

We see the same answer as above. In this example, 0.5 is a constant and m did not change and so is also a constant, so both 0.5 and m cancel in the numerator and denominator. The answer to this question and the first question is the same because we see that A is proportional to r^2 and K is proportional to v^2 so if both v and r increase by 20% both K and A will increase by 44%.

Let's look at another example using $K = 0.5mv^2$. How much does K change if m is reduced to 1/3 of its original value and v is doubled. Writing down what we know:

$$m_{new} = \frac{1}{3}m_{original} \quad v_{new} = 2v_{original}$$

and

$$K_{new} = 0.5m_{new}v_{new}^2 \quad K_{original} = 0.5m_{original}v_{original}^2.$$

Taking the ratio gives

$$\frac{K_{new}}{K_{original}} = \frac{0.5(m_{new})(v_{new})^2}{0.5(m_{original})(v_{original})^2} = \frac{0.5(\frac{1}{3}m_{original})(2v_{original})^2}{0.5(m_{original})(v_{original})^2}$$

.

Canceling out the 0.5, $m_{original}$ and $v_{original}^2$ gives us
$= (\frac{1}{3}) * 2^2 = \frac{4}{3}$

So the new K is 4/3 times larger than the original K.

Be very careful, $(2v_{original})^2 = 4v_{original}^2$. A common mistake is to forget to square the 2

$$(2v_{original})^2 \neq 2v_{original}^2.$$

One last example: If $K = 0.5mv^2$ and if K changes by a factor of 4, and m stays constant, how much does v change? In order to solve this, one must first solve for v. $v{=}\sqrt{\frac{K}{0.5m}}$. Now we solve it exactly the way we solved the above problems

$$\frac{v_{new}}{v_{original}} = \frac{\sqrt{\frac{K_{new}}{0.5m_{new}}}}{\sqrt{\frac{K_{original}}{0.5m_{original}}}}$$

since m does not change this simplifies to

$$\frac{v_{new}}{v_{original}} = \frac{\sqrt{K_{new}}}{\sqrt{K_{original}}} = \sqrt{\frac{K_{new}}{K_{original}}}$$

Since the question states K changes by a factor of 4, we get $K_{new} = 4K_{original}$ so $\frac{v_{new}}{v_{original}} = \sqrt{4}$ or $v_{new} = 2v_{original}$.

Practice problems

1. $A = B$; if B doubles, how much larger is A?
2. $A = B/5$; if B doubles, how much larger is A?
3. $A = 0.3B^2$; if B triples, how much larger is A?
4. $A = 1.2B^3$; if B is halved, how much smaller is A?
5. $A = \frac{1}{B}$; if B doubles, how much smaller is A?
6. $A = \frac{1}{B^2}$; if B doubles, how much smaller is A?
7. $A = \frac{1}{B^2}$; if A doubles, how much smaller is B?
8. $A = BC$; if B doubles and C is halved, how much larger is A?
9. $A = \frac{B}{C^2}$; if B doubles and C doubles, how much smaller is A?
10. $A = \frac{B}{C^2}$; if C doubles and A doubles, how much larger is B?

Answers:

1. Doubles
2. Doubles
3. 9 times larger
4. 1/8 as large
5. Half as large
6. One fourth as large
7. $B = \frac{1}{\sqrt{A}}$ so $B = \frac{1}{\sqrt{2}}$ times smaller
8. No change
9. Half as large
10. 8 times larger