★ C_v & C_p are virtually the same for a liquid

Problem 4 (10 Points):

The coffee purchased at rest stops is often too hot to drink. One way to cool off your coffee is to add ice, but how much ice should you add? Take the initial conditions for the coffee to be $T_0^{cof} = 80\,°\mathrm{C}$ and $V = 400\,\mathrm{ml}$. Take the initial conditions for the ice to be $T_0^{ice} = 0\,°\mathrm{C}$. The final temperature for the coffee and ice that you want to achieve is $T_f = 60\,°\mathrm{C}$. For the following questions assume that the coffee is pure water (a good assumption for most rest stop coffee) and the process is adiabatic with respect to the surroundings. Neglect volume changes of the coffee and ice and any temperature dependence of the heat capacity. The following thermodynamic properties of water may be useful:

$M = 18.0\,\mathrm{g\,mole^{-1}}$, molar mass
$\rho = 1.00\,\mathrm{g/cm^3}$, density
$\Delta H_{fus} = 6.00\,\mathrm{kJ\,mole^{-1}}$, heat of fusion
$C_p = 75.4\,\mathrm{J\,mole^{-1}\,K^{-1}}$, heat capacity of liquid

For parts (a.)-(c.) your answers should be in terms of the variables described here.

a. Find a general (algebraic solution) expression for the mass of ice, m, that is needed to cool the coffee to T_f? **(4 Points)**

b. Calculate, numerically, how many grams of ice you should add to your coffee to lower the temperature to $T_f = 60\,°\mathrm{C}$. **(1 Points)**

c. What is the entropy change of the system (coffee + ice)? Find an algebraic solution. **(3 Points)**

d. What is the entropy change of the surroundings? **(1 Points)**

d) zero - heat released by coffee - absorbed by ice.

e. Is this a thermodynamically reversible process? Explain. **(1 Points)**

e) No, because $S > 0$ so irreversible

a) $Q_{water} = C_p m \Delta T = C_p \rho V (80-60)$

$Q_{ice} = mL + C_p m \Delta T = m_{ice} L + C_p m_{ice}(60-0)$

$$\left.\begin{matrix} \\ \end{matrix}\right\} C_p \rho V 20 = m_{ice}(L + 60 C_p)$$

$$m_{ice} = \frac{C_p \rho V 20}{L + 60 C_p}$$

b) $m_{ice} = \dfrac{\frac{75.4}{18} \cdot 1 \cdot 400 \cdot 20}{\frac{6000}{18} + 60 \cdot \frac{75.4}{18}}$ (note $1\,\mathrm{mL} = 1\,\mathrm{cm}^3$) $\cong 57.3\,\mathrm{g}$

c) $\Delta S_{tot} = \Delta S_{coffee} + \Delta S_{ice\ warm} + \Delta S_{ice\ melt}$ and $\Delta S = \frac{Q}{T}$ and $Q_{H_2O} = mC_p \int dT$

$= C_p m_{coffee} \int_{T_i coffee}^{T_f coffee} \frac{1}{T} dT + C_p m_{ice} \int_{T_i ice}^{T_f ice} \frac{1}{T} dT + \frac{mL}{T_{i ice}}$

$Q_{ice\ warm} = m_{ice} C_p \int dT$

$Q_{ice\ melt} = m_{ice} L$

273° K!

Problem 4 (10 Points):

This problem considers a photon gas. A blackbody cavity can be considered to contain a gas that obeys the equations of state:

$$U = b\ V\ T^4$$

$$P\ V = \tfrac{1}{3}\ U$$

where U is the internal energy, V is the volume, P is the pressure, T is the temperature and $b = 7.56 \times 10^{-16}\ J/(m^3 K^4)$. Note that there is no dependence on N, the number of particles.

a. Show that the fundamental equation for the entropy, S(U,V) is:

$$S(U,V) = \tfrac{4}{3}\ b^{1/4}\ U^{3/4}\ V^{1/4}$$

a) $U = bVT^4$ $\quad dU = TdS - PdV + \sum_i \mu_i dN$

$T = \left(\frac{U}{bV}\right)^{1/4}$ $\quad (dU = TdS)_{\text{const } V,N}$

$\frac{dS}{dU} = \frac{1}{T} = \left(\frac{U}{bV}\right)^{-1/4}$

$\int dS = \int \left(\frac{U}{bV}\right)^{-1/4} dU$

$S = \frac{4}{3} U^{3/4} (bV)^{1/4}$ ✓

(3 Points)

b. The universe can be treated as an expanding electromagnetic cavity at a temperature of T = 2.7 K. Assume the expansion of the universe is isoentropic. What will the temperature of the universe be when it is twice its current size? **(2 Points)**

b) Isoentropic = S is const

$\frac{4}{3} b^{1/4} V^{1/4} U^{3/4} = \text{const}$

$\uparrow$

bVT^4 plug in

$\frac{4}{3} bVT^3 = \text{const}$; $V_2 = 2V_1$

$\cancel{\tfrac{4}{3}}\cancel{b}\cancel{V_1}T_1^3 = \cancel{\tfrac{4}{3}}\cancel{b}(2\cancel{V_1})T_2^3$

$T_1^3 = 2T_2^3$

$T_2 = \frac{1}{2^{1/3}} T_1 = \underline{\underline{2.14\,k}}$

c. What is the pressure associated with the electromagnetic radiation? **(1 Points)**

c) $P = \frac{U}{3V} = \frac{bVT^4}{3V} = \underline{\underline{\frac{1}{3} bT^4}}$

d. What is the Helmholtz potential for this system as a function of U and P? **(3 Points)**

d) $F = E - TS = bVT^4 - T\frac{4}{3}b^{1/4}U^{3/4}V^{1/4} = bVT^4 - \frac{4}{3}Tb^{1/4}(bVT^4)^{3/4}V^{1/4}$

$= bVT^4 - \frac{4}{3}bVT^4 = -\frac{1}{3}U = -PV$ ✓

e. Why is there no dependence upon N in the fundamental equation for the photon gas? **(1 Points)**

e) Because N is not conserved for photons!

Problem 1: (10 Points)

We can use the temperature rise that results from the adiabatic compression of an ideal monoatomic gas to measure the velocity of a bullet. Suppose a piston of mass M can move in a uniform frictionless tube of cross-sectional area A. The piston can only move in the direction of compression. The tube is closed at one end, and the piston is sealed so that no gas can escape. The cylinder is filled with He gas at temperature T_0 and pressure P_0, such that the initial position of the piston is L_0 from the closed end. A bullet of mass m is fired from a gun and strikes the center of the piston. The bullet embeds itself in the piston, causing the piston to move and compress the gas in the tube. The maximum temperature of the gas in the cylinder is T_f. Assume that the piston compresses the gas adiabatically.

a. Find the initial velocity of the bullet, v_0, in terms of the given parameters. **(2 Points)**

b. What is the maximum displacement of the piston, ΔL, in terms of the given parameters? **(2 Points)**

c. What is the maximum final pressure inside the cylinder, P_f, in terms of the given parameters? **(2 Points)**

d. Sketch the acceleration of the piston versus ΔL beginning at the moment the bullet hits the piston. Make sure that the sketch is *qualitatively* accurate. **(2 Points)**

e. Neglecting the exact time that the bullet impacts the piston, at what value of ΔL is the piston at when the magnitude of its acceleration is greatest? **(1 Points)**

f. We assumed that the gas was compressed adiabatically. If heat was lost to the walls of the cylinder, would the resulting value of v_0 be: (1.) too high, (2.) too low, or (3.) unchanged. To receive credit you must explain your answer. **(1 Points)**

a) Inelastic collision, then E cons.

$mv_0 = (M+m)V$

$V = \frac{mv_0}{(M+m)}$

$\Delta KE = 0 - \frac{1}{2}(M+m)V^2$

$= -\frac{(mv_0)^2}{2(M+m)}$

ΔE of monatomic ideal gas

$= \frac{3}{2}Nk\Delta T = \frac{3}{2}Nk(T_f - T_i)$

Set magnitudes equal

$\frac{3}{2}\underset{\uparrow \frac{P_0V_0}{T_0}}{Nk}(T_f - T_0) = \frac{(mv_0)^2}{2(M+m)}$

$$V_0 = \sqrt{\frac{3P_0AL_0(T_f - T_0)(M+m)}{m^2T_0}}$$

b) Use $TV^{\gamma-1} = \text{const}$

$T_0L_0^{2/3} = T_fL_f^{2/3}$

$L_f = \left(\frac{T_0}{T_f}\right)^{3/2}L_0 \quad L_f = L_0 + \Delta L$

$\Delta L = L_0\left[\left(\frac{T_0}{T_f}\right)^{3/2} - 1\right]$ → If we take ΔL as (+) then $L_f = L_0 - \Delta L$

$\Delta L = L_0\left[1 - \left(\frac{T_0}{T_f}\right)^{3/2}\right]$

c) Use $PV^{\gamma} = \text{const}$

$P_f = P_0\left(\frac{V_0}{V_f}\right)^{5/3} = P_0\left(\frac{L_0}{L_f}\right)^{5/3}$

$P_f = P_0\left(\frac{T_f}{T_0}\right)^{5/2}$ by plugging in $L_f = L_0 + \Delta L$

↳ If we take $L_f = L_0 - \Delta L$, we get the same answer

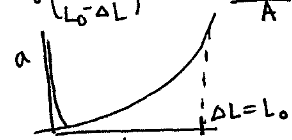
d) Cannot use $W = \int PdV$ since P varies & we don't know how.

Cannot use kinematics since $a \neq \text{constant}$.

$P = \frac{F}{A} = \frac{(M+m)a}{A}$ use $P_f = P_0\left(\frac{L_0}{L_f}\right)^{5/3}$ from $PV^{\gamma} = \text{const}$

$P = P_0\left(\frac{L_0}{L_0 - \Delta L}\right)^{5/3} = a\frac{(M+m)}{A}$

$a = \frac{P_0AL_0^{5/3}}{(M+m)(L_0 - \Delta L)^{5/3}}$

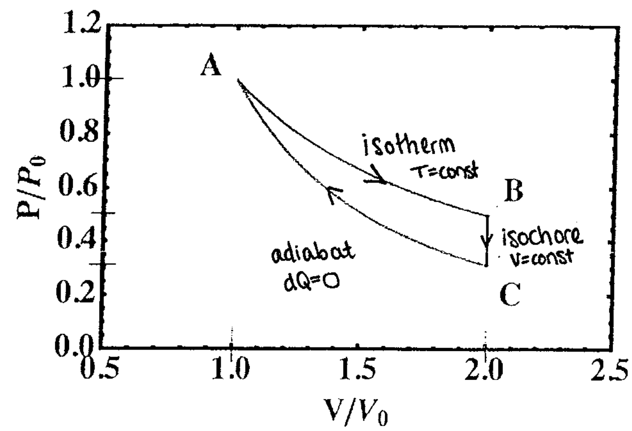


e) When $\Delta L = L_0$ (or fullest compression) $\boxed{|a| \text{ is greatest}}$

f) If some heat is lost to the outside, the same ΔL will stop less velocity. So our original v_0 is higher than what would be stopped.

Statistical Mechanics

4. Consider an ideal monatomic gas used as the working fluid in a thermodynamic cycle. The number of particles is n_0. It follows a cycle consisting of one adiabat, one isochore and one isotherm, as shown below.



(a) Calculate the pressure, temperature, and volume at each corner of the cycle, A, B, and C, expressing your answer in terms of P_0, V_0, n_0 and perhaps R, the ideal gas constant. Note that point A the pressure is P_0 and the volume is V_0. (3pts)

(b) Calculate the work done on the system, the heat into the system and the change in the internal energy of the system for each process step. (4.5pts)

(c) What direction around the cycle must the system follow to be used as a functional heat engine? (1/2pt) Clockwise

(d) What is the efficiency of the cycle, run as an engine? (1pt)

(e) What is the efficiency of an ideal Carnot engine run between reservoirs B and C? (1pt)

a)

	P	V	T
A	$P=P_0$	$V=V_0$	$T=\frac{P_0V_0N_A}{n_0R}$
B	$P=0.5P_0$	$V=2V_0$	$T=\frac{P_0V_0N_A}{n_0R}$
C	$P=0.31P_0$	$V=2V_0$	$T=\frac{P_0V_0N_A}{n_0R}\,0.62$

$P_BV_B=P_AV_A$

$P_B=\frac{P_AV_A}{V_B}=0.5P_0$

$P_CV_C^{\gamma}=P_AV_A^{\gamma}$ $\quad \gamma=\frac{f+2}{f}=\frac{5}{3}$

$P_C=\frac{P_0V_0^{5/3}}{(2V_0)^{5/3}}$

$\boxed{\Delta U=Q+W}$

b)

	$W_{\text{on sys}}$	Q_h	ΔU
CA adiabat	$0.6P_0V_0$	0	$0.6P_0V_0$
BC isochore	0	$-0.6P_0V_0$	$-0.6P_0V_0$
AB isotherm	$-P_0V_0\ln 2$	$P_0V_0\ln 2$	0

isotherm: $W=\int P\,dV$ $\quad P=\frac{n_0RT}{N_A V}$

$W_{AB}=\frac{n_0RT}{N_A}\int\frac{1}{V}dV$

$\Delta U=\frac{3}{2}Nk\Delta T$

$\Delta U_{BC}=\frac{3}{2}Nk(T_C-T_B)$

$\Delta U_{CA}=\frac{3}{2}Nk(T_A-T_C)$

d)

$\eta=\frac{W_{net}}{Q_h}$

$=\frac{P_0V_0\ln 2-0.6P_0V_0}{\ln 2\,P_0V_0}$

$\cong 9.1\%$

or $\eta=1-\frac{Q_c}{Q_h}\cong 9.1\%$ ✓

e)

$\eta=1-\frac{T_L}{T_H}$

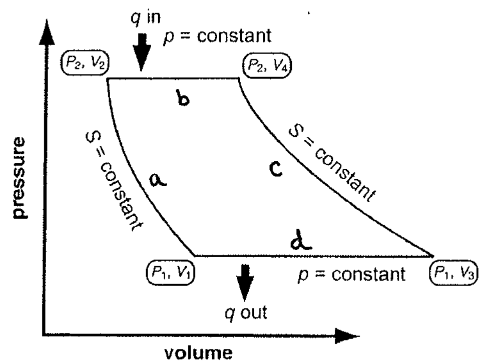
$=1-\frac{(PV)_C}{(PV)_B}$

$\cong 37\%$

5-09

Statistical Mechanics

4. The gas turbine (jet engine) can be modeled as a Brayton cycle. Below is the P-V diagram for this process.



Assume that the working fluid is an ideal monatomic gas.

(a) Calculate the work done by the gas on each step in the cycle. (3 pts.)

(b) Find the heat for each step in the cycle. (3 pts.)

(c) Find the efficiency of this engine. Your answer should be in terms of the pressures (P_1 and P_2) and the volumes (V_1, V_2, V_3, and V_4). (3 pts.)

(d) To produce work, which way does the cycle operate? Clockwise or counter clockwise? (1 pt.) clockwise

a) $W = W_a + W_b + W_c + W_d$

$W_b = P_2(V_4 - V_2)$ $\quad dW_a = dE_a$ since $dQ = 0$ $\quad$ Similarly, $W_c = \frac{3}{2}(P_1V_3 - P_2V_4)$

$W_d = P_1(V_1 - V_3)$ $\quad dE = \frac{3}{2}NkdT$

So $W_a = \frac{3}{2}Nk(T_2 - T_1) = \frac{3}{2}(P_2V_2 - P_1V_1)$

$PV = NkT$

$T = \frac{PV}{Nk}$

So: $W = \frac{3}{2}(P_2V_2 - P_1V_1 + P_1V_3 - P_2V_4) + P_2V_4 - P_2V_2 + P_1V_1 - P_1V_3$

$$W = \frac{1}{2}(P_2V_2 - P_1V_1 + P_1V_3 - P_2V_4)$$

b) $Q_b = \frac{C_p}{nk}(P_2V_4 - P_2V_2)$ $\quad Q_a = Q_c = 0$

$Q_d = C_p\Delta T_d = \frac{C_p}{nk}(P_1V_1 - P_1V_3)$

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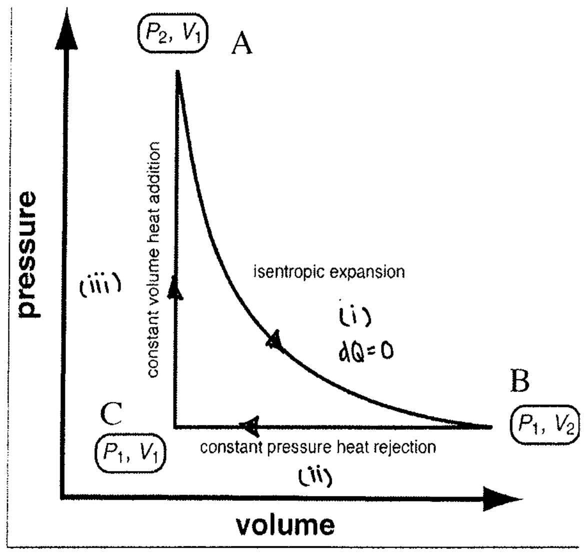
c) $$\eta = \frac{W}{Q_h} = \frac{\frac{1}{2}(P_2V_2 - P_1V_1 + P_1V_3 - P_2V_4)}{\frac{C_p}{nk}(P_2V_4 - P_2V_2)} = 1 - \frac{Q_c}{Q_h} = 1 - \frac{Q_d}{Q_b} = 1 - \frac{P_1(V_1 - V_3)}{P_2(V_4 - V_2)}$$

Statistical Mechanics

4. **Heat Engines:** A pulse jet operates under a Lenoir cycle. This consists of an adiabat, an isobar, and an isochore, as shown.



Assuming that the working fluid is an ideal 3D monoatomic gas of N particles:

(a) Find the work done in one complete cycle. (3 points)

(b) Find the heat exchanged in each step in the cycle. (3 points)

(c) Find the efficiency of the engine. Express your answer in terms of pressures and volu mes. (3 points)

(d) To produce work, should the engine cycle operate clockwise ($A \to B \to C \to A$) or counterclockwise ($A \to C \to B \to A$) ? (1 point) } Clockwise to make (+) area under curve

$PV = nRT$

$T = \frac{PV}{nR}$

a) $W = W^{i} + W^{ii} + \overset{0,\text{ const vol}}{\cancel{W^{iii}}} = \int_A^B P\,dV + P_1(V_1 - V_2) \Rightarrow$

PV^{γ} const from $A \to B$

$P = \frac{P_1 V_2^{\gamma}}{V^{\gamma}} \to \int P\,dV = \frac{P_1 V_2^{\gamma}}{-\gamma+1}\left[V_2^{-\gamma+1} - V_1^{-\gamma+1}\right]$

$\Big\}\; W = P_1(V_1 - V_2) + \frac{P_1 V_2^{\gamma}}{1-\gamma}\left[V_2^{-\gamma+1} - V_1^{-\gamma+1}\right]$

Better $\to$ OR: $W = |Q_h| - |Q_c| = C_v(T_A - T_C) - C_p(T_C - T_B) = \frac{C_v}{nR}\left[P_2V_1 - P_1V_1\right] - \frac{C_p}{nR}\left[P_1V_1 - P_1V_2\right] = \frac{C_v}{nR}V_1(P_2 - P_1) - \frac{C_pP_1}{nR}(V_1 - V_2)$

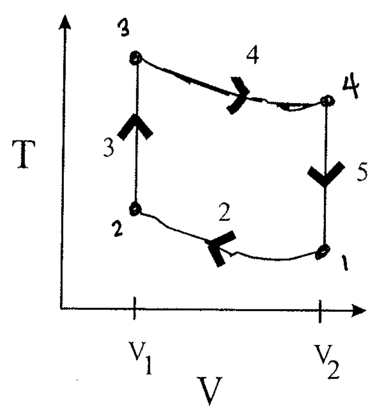
b) $Q^{i} = 0,\ Q^{ii} = \frac{C_p}{nR}\left[P_1V_1 - P_1V_2\right],\ Q^{iii} = \frac{C_vV_1}{nR}(P_2 - P_1)$

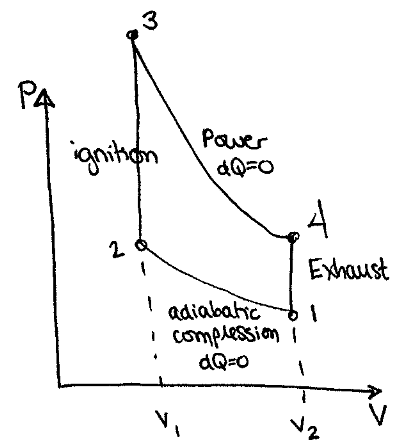
c) $\eta = \frac{W}{Q_h} = \frac{Q_h - Q_c}{Q_h} = 1 - \frac{Q_c}{Q_h} = 1 - \frac{C_pP_1\,(V_1 - V_2)}{C_vV_1\,(P_2 - P_1)} = 1 + \gamma\,\frac{(1 - V_2/V_1)}{(1 - P_2/P_1)}$

d) Clockwise

Problem 5 (10 Points):

The Otto cycle is shown in the figure. Stages 2 and 4 are adiabatic, reversible expansion and compression. Stages 3 and 5 are constant volume heating and cooling. Assume this is for an ideal gas.





a. Write down the efficiency, η, in terms of the work, W, and the added heat, Q. **(2 Points)**

a) $\rightarrow \eta = \frac{W}{Q_h}$ by definition, Q_h = heat absorbed

b. During which stage or stages is heat added? **(1 Points)**

b) $\rightarrow$ during 3

c. Calculate the work, W, in terms of the heat capacity at constant volume, C_V and the temperature change. **(2 Points)**

$\rightarrow$ c) $W = Q_h - Q_c = C_v m\left[(T_3 - T_2) + (T_1 - T_4)\right]$

d. Calculate the heat added, Q, in terms of C_V and the temperature change. **(2 Points)**

$\rightarrow$ d) $Q_h = C_v m (T_3 - T_2)$

e. Show that the efficiency is η=1-$\frac{V_1}{V_2}^{\gamma-1}$, where γ=$\frac{C_p}{C_V}$. **(3 Points)**

e) $\eta = \dfrac{Q_h - Q_c = C_v m(T_3 - T_2) + C_v m(T_1 - T_4)}{C_v m (T_3 - T_2)} = 1 + \dfrac{T_4 - T_1}{T_2 - T_3} = 1 - \dfrac{T_1 - T_4}{T_2 - T_3} = 1 - \left(\dfrac{T_1}{T_2}\right)\left(\dfrac{T_4/T_1 - 1}{T_3/T_2 - 1}\right)$

Show $\dfrac{T_4}{T_1} = \dfrac{T_3}{T_2}$: $T_3 V_3^{\gamma-1} = T_4 V_4^{\gamma-1}$, $\dfrac{T_3}{T_4} = \left(\dfrac{V_2}{V_1}\right)^{\gamma-1}$; $T_1 V_2^{\gamma-1} = T_2 V_1^{\gamma-1}$, $\dfrac{T_1}{T_2} = \left(\dfrac{V_1}{V_2}\right)^{\gamma-1}$ So $\dfrac{T_3}{T_4} = \dfrac{T_2}{T_1} \rightarrow \dfrac{T_4}{T_1} = \dfrac{T_3}{T_2}$ So that term cancels ✓

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And $\dfrac{T_1}{T_2} = \left(\dfrac{V_1}{V_2}\right)^{\gamma-1}$ from above. So $\eta = 1 - \left(\dfrac{T_1}{T_2}\right) = 1 - \left(\dfrac{V_1}{V_2}\right)^{\gamma-1}$ ✓