5. Consider the quantum mechanical linear rotator. It has energy levels

$$E_J = \frac{\hbar^2}{2I} J(J+1)$$ "The" rotator, so assume $N=1$

where I is the moment of inertia and J is the angular momentum quantum number, $J = 0, 1, 2, \ldots$. Each energy level is $(2J+1)$-fold degenerate.

(a) In the low temperature limit ($\hbar^2/2I \gg kT$) determine approximate expressions for:

i. The rotation partition function. (2pts)

ii. The internal energy. (1pt)

iii. The specific heat. (1pt)

(b) In the high temperature limit ($\hbar^2/2I \ll kT$) determine approximate expressions for:

i. The rotation partition function. (2pt)

ii. The internal energy. (1pt)

iii. The specific heat. (1pt)

(c) How do the quantum results compare with the equipartition theorem for a classical rotator with two transverse degrees of freedom? (2pts)

$$\mathcal{Z} = \sum_J (2J+1) e^{-\beta\hbar^2/2I\, J(J+1)} = Z \text{ for } N=1$$

a) In the low temp limit: $T \to 0$, $\beta \to \infty$, $e^{-\beta \text{stuff}}$ is small, so keep just 1st few terms.

i) $Z = \mathcal{Z} = 1 + 3e^{-\beta\hbar^2/I} + 5e^{-\beta\hbar^2 3/I} + \ldots$

ii) $E = -\frac{\partial}{\partial\beta}\ln Z = \frac{-1\left(3e^{-\beta\hbar^2/I}\right)\left(-\frac{\hbar^2}{I}\right)}{1+3e^{-\beta\hbar^2/I}} = \frac{3\hbar^2/I}{e^{\beta\hbar^2/I}+3}$ now $e^{\beta\hbar^2/I} \gg 3$ so $E \cong\!> \frac{3\hbar^2}{I} e^{-\beta\hbar^2/I}$

iii) $C_v = \frac{\partial E}{\partial T} = -k\beta^2 \frac{\partial E}{\partial\beta} = \frac{-k\beta^2\, 3\hbar^2/I}{\left(e^{\beta\hbar^2/I}+3\right)^2}\left(e^{\beta\hbar^2/I}\right)\left(\frac{\hbar^2}{I}\right) = \frac{-3k\beta^2\hbar^4/I^2\, e^{\beta\hbar^2/I}}{\left(e^{\beta\hbar^2/I}+3\right)^2} \cong -3k\beta^2\hbar^4/I^2\, e^{-\beta\hbar^2/I}$

b) In the high temp limit: $T \to \infty$, $\beta \to 0$, β(stuff) is small so energy levels are close together. Use integral.

i) $\mathcal{Z} \cong \int_0^\infty (2J+1) e^{-\beta\hbar^2/2I\, J(J+1)}\, dJ = \int e^{-\alpha u}\, du \to \mathcal{Z} \cong -\frac{2I}{\beta\hbar^2}\left[e^{-\infty} - 1\right] = Z = \frac{2I}{\beta\hbar^2}$

ii) $E = -\frac{\partial}{\partial\beta}\ln Z = -\frac{\partial}{\partial\beta}\left[\ln\left(\frac{+2I}{\beta\hbar^2}\right)\right] = -\frac{\beta\hbar^2}{2I}\left(\frac{-2I}{\beta^2\hbar^2}\right) = \frac{1}{\beta} = kT$ ✓ matches classical case

iii) $C_v = -k\beta^2\frac{\partial E}{\partial\beta} = -k\beta^2\left[\frac{\partial}{\partial\beta}\left(\frac{1}{\beta}\right)\right] = -k\beta^2\left(-\frac{1}{\beta^2}\right) = k$ ✓ matches classical case

c) Quantum effects should dominate at low T & Classical at high T. This is what we see. E for a 2D classical rotator is $\frac{2}{2}NkT$ which is what we found for high T. C_v also matches.

6. Consider the "bogon," a spin 5/2 fermion with the charge of an electron but with a dispersion relationship

$$E = cp^3.$$

where $p \equiv |\vec{p}|$ Assume that your bogons are confined in a three dimensional sample and are non-ineracting.

(a) Working in the grand canonical ensemble, determine the density, $\rho = \langle N \rangle / V$, as a function of the chemical potential, μ (or the fugacity, $z \equiv e^{\beta\mu}$), T, and V. (3pts)

(b) What is the bogonic Fermi energy (μ at $T = 0$) as a function of their density? (3pts) (*Hint:* This should not involve any complicated integrals).

(c) Derive a series expansion in z for the grand canonical free entropy, $\Xi = \frac{PV}{kT} = \log \mathcal{Z}$, where $\mathcal{Z}$ is the grand canonical partition function. (4pts)

a) $\sum_{N=0}^{\infty} \sum_{\{n_{i\sigma}\}} e^{N\beta\mu - \beta E_{i,\sigma} \sum_{i\sigma} n_{i\sigma}}$ summing over states & spins!

$$Q = \sum_{N=0}^{\infty} \sum_{\{n_{i\sigma}\}} e^{\sum_{i\sigma} n_{i\sigma} \beta(\mu - E_{i\sigma})} = \prod_i \left(\sum_{n_i=0}^{\infty} e^{-n_i \beta (E_i - \mu)} \right)^{2S+1} \text{ where } S = 5/2 \; = \prod_i \left(\sum_{n_i=0}^{\infty} e^{-\beta(E_i - \mu)} \right)^{n_i 6}$$

Now $n_i = 0$ or 1 for fermions:

$$Q = \prod_i \left(1 + e^{-\beta(E_i - \mu)} \right)^6 \quad \mathcal{Y} = -\frac{1}{\beta} \ln Q = -\frac{1}{\beta} \sum_i 6 \ln\left(1 + e^{-\beta(E_i - \mu)}\right) \quad N = -\frac{\partial \mathcal{Y}}{\partial \mu} = \sum_i \frac{6}{\beta} \frac{1}{1 + e^{+\beta(E_i - \mu)}} (\beta) = \sum_i f(E_i) \checkmark$$

Now our $\sum_i$ becomes a sum over a "really big box" - goes to an integral.

$$N \cong \int \frac{d^3x d^3p}{(2\pi\hbar)^3} \frac{6}{1 + e^{\beta(E-\mu)}} \cong \frac{4\pi V 6}{(2\pi\hbar)^3} \int \frac{p^2 dp}{e^{\beta(cp^3 - \mu)} + 1} \quad \text{so} \quad \rho = \frac{\langle N \rangle}{V} = \underline{\underline{\frac{24\pi}{(2\pi\hbar)^3} \int \frac{p^2 dp}{e^{\beta(cp^3 - \mu)} + 1}}}$$

b) At $T=0$, $\mu = \epsilon_f$, $f(\epsilon) = \delta(\epsilon_f - \epsilon)$ $\quad E = cp^3 \rightarrow dE = 3p^2 dp c \rightarrow p^2 dp = \frac{dE}{3c}$

$$\rho = \frac{3}{\pi^2 \hbar^3} \int_0^{\infty} \Theta(\epsilon_f - \epsilon) \frac{d\epsilon}{3c} = \frac{\cancel{\beta}\, \epsilon_F}{\cancel{\beta} c \pi^2 \hbar^3} \quad \text{so} \quad \underline{\underline{\epsilon_F = \rho c \pi^2 \hbar^3 = \mu}}$$

c) $$\psi = \ln Q = \sum_i \ln\left(1 + e^{-\beta(E_i - \mu)}\right)^6 = 6 \int \frac{d^3x d^3p}{h^3} \ln\left(1 + z e^{-\beta E_i}\right) = \frac{6(4\pi)V}{(2\pi\hbar)^3} \int_0^{\infty} p^2 dp \ln\left(1 + z e^{-\beta c p^3}\right)$$

$E = cp^3$

Use $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ for $-1 < x \leq +1$ works since z is assumed to be small.

$$\psi = \frac{3V}{3c\pi^2\hbar^3} \int_0^{\infty} \ln(1 + z e^{-\beta E}) dE = \frac{V}{\pi^2 \hbar^3 c} \int_0^{\infty} \left(z e^{-\beta E} - \frac{(z^2 e^{-2\beta E})}{2} + \frac{(z^3 e^{-3\beta E})}{3} - \cdots \right) dE$$

Could have left in terms of p — is just more elegant ☺

6. A black body may be thought of as a system of harmonic oscillators possessing all possible frequencies—equivalently, it is a system of photons governed by the Bose-Einstein distribution.

   (a) Calculate the average energy $u(\nu)$ of a quantum harmonic oscillator of frequency ν at temperature T where the allowed energies of the oscillator are:
   $$E(n) = h\nu n$$
   and we have ignored the zero-point energy. (3 points)

   (b) The number of oscillators per unit phase space is $2\,d^3q\,d^3p/h^3$, where the factor of 2 comes from the two transverse polarization states of the photon. Calculate the total energy of the black body
   $$U = 2\int \frac{d^3q\,d^3p}{h^3}u(\nu)$$
   in terms of a single dimensionless integral. This is the famous Planck formula. [Use the relativistic relation between frequency and momentum for photons, $h\nu = pc$.] (3 points)

   (c) Derive the Stefan-Boltzmann law, $u = aT^4$, and compute the constant a using the formula
   $$\int_0^\infty dx\frac{x^{n-1}}{e^x - 1} = \zeta(n)\Gamma(n),$$
   where $\zeta(n)$ is the Riemann zeta function, and $\Gamma(n)$ is the gamma function . Your answer will be in terms of mathematical and physical constants. (4 points)

a) For one oscillator: $\mathcal{Z} = Z = \sum_n \left(e^{-\beta h\nu}\right)^n = \frac{1}{1-e^{-\beta h\nu}}$

$$E = -\frac{\partial}{\partial\beta}\ln Z = \frac{\partial}{\partial\beta}\ln\left(1-e^{-\beta h\nu}\right) = \frac{1}{1-e^{-\beta h\nu}}\left(e^{-\beta h\nu}\right)(-h\nu) = \underline{\underline{\frac{h\nu}{e^{\beta h\nu}-1}}} = f(h\nu)E(h\nu) \ \checkmark$$

b) $$U = 2\int\frac{d^3x\,d^3p}{h^3}\frac{h\nu}{e^{\beta h\nu}-1} = \frac{2V4\pi}{h^3c}\int\frac{p^3dp}{e^{\beta pc}-1}$$ using $h\nu = pc = E$, $dE = c\,dp$ $$U = \frac{8\pi V}{h^3c^3}\int\frac{E^3dE}{e^{\beta E}-1}$$ then use $E = \frac{x}{\beta} = kTx$ so that x is dimensionless

$$\underline{\underline{U = \frac{8\pi V}{h^3c^3\beta^4}\int\frac{x^3dx}{e^x-1}}}$$

c) $$u = \frac{U}{V} = \frac{8\pi}{h^3c^3\beta^4}\int\frac{x^3dx}{e^x-1} = \frac{8\pi}{h^3c^3\beta^4}\zeta(4)\Gamma(4) = \underbrace{\frac{8\pi k^4}{h^3c^3}\zeta(4)\Gamma(4)}_{a}T^4$$ so $$\underline{\underline{a = \frac{8\pi k^4}{h^3c^3}\zeta(4)\Gamma(4)}}$$

also note $\zeta(4)\Gamma(4) = \frac{\pi^4}{15}$ (simpler answer)

Problem 5 (10 Points):

Consider a one dimensional ideal gas of electrons as a model for the conduction electrons in a one dimensional wire.

a. Determine the density of states g(E) for the one dimensional non-interacting electron system confined to a length, L. **(3 Points)**

b. What is the Fermi energy for this system? **(2 Points)**

c. What is the root mean square velocity of the electrons at $T = 0\,^{\circ}$K? **(3 Points)**

d. What is the entropy of the electrons at $T = 0\,^{\circ}$K? Justify your answer. **(2 Points)**

a) $g(E) = \int dk\, D_k\, \delta(E-E)$ for one dim, n=1 where $D_k = 2\left(\frac{L}{2\pi}\right)^n$

$= \int dk\, 2\left(\frac{L}{2\pi}\right)\delta(E-E)$ use $E = \frac{\hbar^2 k^2}{2m} \rightarrow k = \sqrt{\frac{2mE}{\hbar^2}} \rightarrow dk = \frac{1}{2}\sqrt{\frac{2m}{E\hbar^2}}\, dE$

$= \frac{L}{\pi}\int \frac{1}{2}\sqrt{\frac{2m}{\hbar^2 E}}\, dE\, \delta(E-E) = \underline{\underline{\frac{L}{2\pi}\sqrt{\frac{2m}{\hbar^2 E}}}}$

b) At T=0°K, $N = \int_0^{E_f} g(E)\,dE = \int_0^{E_f} \frac{L}{2\pi}\sqrt{\frac{2m}{\hbar^2 E}}\, dE = \frac{L}{2\pi}\sqrt{\frac{2m}{\hbar^2}}\, 2\sqrt{E}\Big|_0^{E_f} = \frac{L}{\pi}\sqrt{\frac{2mE_f}{\hbar^2}} = N$

$$E_f = \left(\frac{N\pi\hbar}{L}\right)^2 \frac{1}{2m}$$

c) At T=0°K, $\sqrt{V_{avg}^2} \approx V_{rms}$ (not always true!)

Use $E = \frac{1}{2}mv^2$ & $E = \int_0^{E_f} D(E)\,E\,f(E)\,dE = \frac{L}{2\pi\hbar}\int \frac{\sqrt{2m}}{\sqrt{E}}\, E\, \frac{1}{e^{\beta(E-\mu)}+1}\, dE$

$E = \int_0^{E_f} \frac{L\sqrt{2m}}{2\pi\,\hbar\sqrt{E}}\, E\,dE = \frac{L}{2\pi}\frac{\sqrt{2m}}{\hbar}\left(\frac{2}{3}E^{3/2}\right)\Big|_0^{E_f} = \frac{L\sqrt{2m}}{3\pi\hbar}E_f^{3/2} = \frac{L\sqrt{2m}}{3\pi\hbar}\left(\frac{N\pi\hbar}{L}\right)^3\frac{1}{(2m)^{3/2}} = \frac{N^3(\pi\hbar)^2}{3L^2\, 2m} = \frac{N^3(\pi\hbar)^2}{6mL^2}$

$E = \frac{1}{2}mv^2 = \frac{N^3(\pi\hbar)^2}{6mL^2} \rightarrow \underline{\underline{V = \frac{N^{3/2}\pi\hbar}{\sqrt{3}\,mL}}}$

Another way to get V: $\langle V^2\rangle = \int_0^{E_f} V^2 g(\epsilon) f(\epsilon)\, d\epsilon$ using $f(\epsilon) \rightarrow 1$ at T=0

$V^2 = \frac{2E}{m}$ from $E = \frac{1}{2}mv^2$

$g(\epsilon)$ from earlier.

d) $S \rightarrow 0$ at $T \rightarrow 0$. Even though the electrons move around, they do not change state.

Problem 6 (10 Points):

γ = boson
e^- = fermion

The following questions refer to a stream of photons in equilibrium at temperature T (thermal light - say from a light bulb) incident on a perfect detector which detects (counts) all the particles that hit it. Your final answers should be in terms of the mean particle number.

a. Given $\bar{n}_s$ photons are counted on average in time t, calculate the variance in the photon number n_s, $\overline{(\Delta n_s)^2}$. **(2 Points)**

b. Calculate the fractional fluctuation of the detector signal defined as the square root of the variance divided by the mean photon number, $\bar{n}_s$, squared, $\sqrt{\overline{(\Delta n_s)^2}/\bar{n}_s^2}$. This is the inverse of the signal to noise ratio. **(2 Points)**

The following questions refer to a stream of electrons in equilibrium at temperature T incident on a detector which detects (counts) all the particles that hit it. Again, your final answers should be in terms of the mean particle number.

c. Given $\bar{n}_e$ electrons are counted on average in time t, calculate the variance in the electron number n_e, $\overline{(\Delta n_e)^2}$. **(2 Points)**

d. Calculate the fractional fluctuation of the detector signal defined as the square root of the variance divided by the mean electron number, $\bar{n}_e$, squared, $\sqrt{\overline{(\Delta n_e)^2}/\bar{n}_e^2}$. **(2 Points)**

e. Compare the two results. Are the results the same or different? Do the counts detected clump (bunch) or anti-clump (anti-bunch)? Why? **(2 Points)**

a) $\overline{(\Delta n_s)^2} = \langle (n_s - \langle n_s \rangle)^2 \rangle = \langle n_s^2 - 2n_s\langle n_s\rangle + \langle n_s\rangle^2 \rangle = \langle n_s^2\rangle - 2\langle n_s\rangle^2 + \langle n_s\rangle^2$

$= \langle n_s^2\rangle - \langle n_s\rangle^2$

$\mathcal{G} = E - TS - \mu N$

$E = TS - pV + \mu N$

$\mathcal{G} = -pV = -kT \ln Q$

so $\ln Q = \frac{pV}{kT}$

$pV = \frac{1}{\beta} \ln Q = -\mathcal{G}$

$\langle E \rangle = \int E g(E) f(E) dE$

$g(E) = \int d\vec{k}\, \delta(E - E_k) \left(\frac{L}{2\pi}\right)^d$

$= \frac{V 4\pi}{(2\pi)^3} \int k^2 dk\, \delta(E - E_k)$ $\quad k = \frac{\sqrt{2mE_k}}{\hbar} \quad dk = \frac{dE}{2\sqrt{E}}$

$= \frac{V}{2\pi^2} \int \frac{2mE_k}{\hbar^2} \frac{dE}{2\sqrt{E_k}} \frac{\sqrt{2m}}{\hbar} \delta(E - E_k)$

$= \frac{V (2m)^{3/2}}{h^2 \hbar} \sqrt{E}$

$\frac{\langle E \rangle}{V} = \frac{(2m)^{3/2}}{h^2 \hbar} \int \frac{\sqrt{E}\, E}{e^{\beta(E-\mu)} \mp 1} dE$

$\frac{u}{V} = \frac{(2m)^{3/2} 2\pi E^{3/2}}{h^3 (e^{\beta(E-\mu)} \mp 1)}$

spin zero Bose gas in two dimensions.
ven by:

$= \hbar^2 k^2 / 2m$

n. Assume your system is confined to
side.

for the grand canonical free energy
states. Do not evaluate the sum. (1

rticles in the system as a function of

$N(T, V, \mu)$ in the limit $T \to 0$. What
sibility of a Bose-Einstein transition

qual to the energy density, so that
t have to do any sums over states -
holds using analytic expressions for
stem). (3 pts.)

a) $Q = \sum_{N=0}^{\infty} e^{\beta \mu N}$ $\quad e^{-n_k \beta (E_k - \mu)})$ now we have unrestricted n_k, so use our trusty expansion to $\frac{1}{1-x}$

$Q = \prod_k \frac{1}{1 - e^{-f}}$ $\quad -\mu))$ where $E_k = \frac{\hbar^2 k^2}{2m}$

b) $N = -\frac{\partial \mathcal{G}}{\partial \mu}$ $\quad pV = \frac{1}{\beta} \ln Q = \sum_k -\frac{1}{\beta} \ln\left(1 - e^{-\beta(E_k - \mu)}\right)$

akes sense, his is our iist f'n $f(E_k)$ or bosons!

Now we sum over a really big box - so $\sum_k \Rightarrow \frac{L^2}{(2\pi)^2} \int d^2k$ for 2D case.

$N = \frac{A 2\pi}{(2\pi)^2}$

tegic u substitution: $u = 1 - e^{-\beta(E_k - \mu)}$ $\quad du = -e^{-\beta(E_k - \mu)} \left(-\frac{\beta \hbar^2}{2m} 2k\, dk\right)$

so $e^{-\beta(E_k - \mu)} k\, dk = -\frac{2m}{2\beta \hbar^2} du$

$= +\frac{Am}{2\pi \beta \hbar^2} \ln\left(1 - e^{-\beta(E_k - \mu)}\right)\Big|_{k=0}^{k=\infty} = +\frac{Am}{2\pi \beta \hbar^2}\left[\ln(1-0) - \ln(1 - e^{\beta\mu})\right] = \frac{Am}{2\pi \beta \hbar^2} \ln\left(\frac{1}{1 - e^{\beta\mu}}\right)$

c) Take limit of N for $T \to 0$ & $\mu = 0$. If $N \to \infty$ then no condensate!

$N = \frac{Am}{2\pi \beta \hbar^2} \ln\left(\frac{e^{-\beta\mu}}{e^{-\beta\mu} - 1}\right) = \frac{Am}{2\pi \hbar^2 \beta} \ln\left(\frac{1}{1 - e^{\beta\mu}}\right) \cong \frac{Am}{\beta 2\pi \hbar^2} \ln\left(\frac{1}{\infty}\right) \Rightarrow +\infty$ undefined, so no condensate. Shd be the case for a 2D Bose gas. ✓

set $\mu = 0$

BETTER $\Rightarrow$ Could also solve for μ to see if $\mu = E_0$ when $T = 0$. $\mu = kT \ln\left(1 - e^{-\frac{2\pi \beta \hbar^2 N}{Am}}\right)$ ✓ works so no condensate

μ shd equal the ground state energy when $T \to 0$

* Key concept: Convert to COM frame. Have p_{cm}, p_{sep}, & x_{sep}
Vibrations are small compared to length of "box", so x_{sep} goes $-\infty$ to ∞.

5. Consider a gas of N non-interacting **one dimensional** diatomic molecules enclosed in a box of "volume" L (actually, just a length) at temperature T.

   (a) The classical energy for a single molecule is:

   $$E(p_1, p_2, x_1, x_2) = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{1}{2}K(x_1 - x_2)^2$$

   where p_1 and p_2 are the classical momenta of the atoms in one diatomic molecule, x_1 and x_2 are their classical positions, and K is the spring constant. Calculate the specific heat for the gas. (You should assume that $KL^2/2 >> k_B T$, where k_B is Boltzmann's constant.) (4 points).

   (b) In the quantum limit the energy levels of the molecule are discrete. In a semiclassical approach we can write the energy of one molecule as:

   $$E(P, n) = \frac{P^2}{4m} + \hbar\omega(n + \frac{1}{2})$$

   where P is the momentum of the diatomic molecule (of mass $2m$), and ω is the natural frequency of the oscillator, and n is a non-negative integer ($n \geq 0$). Calculate the specific heat. (4 points).

   (c) Calculate the high and low temperature limits of your result in (b), and explain how they relate to the result of (a). (2 points)

a) Convert to COM coordinates.

$$KE = \frac{1}{2}m\left(2\dot{x}_{cm}^2 + \frac{\dot{x}_{sep}^2}{2}\right)$$

$$PE = \frac{1}{2}kx_{sep}^2$$

$$H = \frac{p_{cm}^2}{4m} + \frac{p_{sep}^2}{m} + \frac{1}{2}k(x_{sep})^2$$

$$z = \frac{1}{h^2}\int_{-\infty}^{\infty} e^{-\frac{\beta p_{cm}^2}{4m}}\,dp_{cm}\int_{-\infty}^{\infty} e^{-\frac{\beta p_{sep}^2}{m}}\int_{-\infty}^{\infty} e^{-\frac{\beta k}{2}x_{sep}^2}\,dx_{sep} \quad \text{times} \int_0^L dx$$

$$Z = z^N = \left(\frac{Lm\pi^{3/2}2^{3/2}}{h^2 k^{1/2}\beta^{3/2}}\right)^N$$

$$C_V = k\beta^2\frac{\partial^2}{\partial\beta^2}\ln Z = \underline{\underline{\frac{3}{2}Nk}} \checkmark$$

b) $$z = \int_{-\infty}^{\infty} e^{-\frac{\beta p^2}{4m}}\frac{dp}{2\pi\hbar}\sum_n e^{-\beta\hbar\omega n}\left(e^{-\frac{\beta\hbar\omega}{2}}\right) = \sqrt{\frac{4m\pi}{\beta}}\frac{1}{2\pi\hbar}e^{-\frac{\beta\hbar\omega}{2}}\frac{1}{1-e^{-\beta\hbar\omega}} \text{ and } Z = z^N$$

$$C_V = k\beta^2\frac{\partial^2}{\partial\beta^2}\ln Z = Nk\beta^2\frac{\partial^2}{\partial\beta^2}\left[\frac{1}{2}\ln 4m\pi - \frac{1}{2}\ln\beta - \ln 2\pi\hbar - \frac{\beta\hbar\omega}{2} - \ln(1-e^{-\beta\hbar\omega})\right] = Nk\beta^2\frac{\partial}{\partial\beta}\left[\frac{-1}{2\beta} - \frac{\hbar\omega}{2} - \underbrace{\frac{1(-e^{-\beta\hbar\omega})(-\hbar\omega)}{1-e^{-\beta\hbar\omega}}}_{\frac{-\hbar\omega}{e^{\beta\hbar\omega}-1}}\right]$$

$$= Nk\beta^2\left[\frac{1}{2\beta^2} \quad \frac{-\hbar\omega(-1)(e^{\beta\hbar\omega})(\hbar\omega)}{(e^{\beta\hbar\omega}-1)^2}\right] = \underline{\underline{\frac{1}{2}Nk + \frac{\hbar^2\omega^2 e^{\beta\hbar\omega}}{(e^{\beta\hbar\omega}-1)^2}Nk\beta^2}}$$

c) $T \to 0$, $\beta \to \infty$, $e^{\beta\hbar\omega} >> 1$

$$C_V \cong \frac{1}{2}Nk + \frac{\hbar^2\omega^2 \cancel{e^{\beta\hbar\omega}}}{{}_{\wedge}\, e^{\cancel{2}\beta\hbar\omega}}\,Nk\beta^2 = \underline{\underline{\frac{1}{2}Nk + \hbar^2\omega^2 e^{-\beta\hbar\omega}Nk\beta^2}}$$

qm result dominates

$T \to \infty$, $\beta \to 0$, so $e^{\beta\hbar\omega} \approx 1 + \beta\hbar\omega$

$$C_V \cong \frac{1}{2}Nk + {}_{\wedge}^{Nk\beta^2}\frac{\hbar^2\omega^2 e^{\beta\hbar\omega}}{(1+\beta\hbar\omega-1)^2} = \frac{1}{2}Nk + Nke^{\beta\hbar\omega} = \frac{1}{2}Nk + Nk(1+\beta\hbar\omega)$$

$$\cong \underline{\underline{\frac{3}{2}Nk + Nk\beta\hbar\omega}}$$

Classical effects dominate

6. Fermions:

(a) Show that for any non-interacting spin 1/2 fermionic system with chemical potential μ, the probability of occupying a single particle state with energy $\mu + \delta$ is the same as finding a state vacant at an energy $\mu - \delta$. (2 points)

(b) Consider non-interacting fermions that come in two types of energy states:

$$E_{\pm}(\vec{k}) = \pm\sqrt{m^2c^4 + \hbar^2k^2c^2}$$

At zero temperature all the states with negative energy (all states with energy $E_-(\vec{k})$) are occupied[1] and all positive energy states are empty, and that $\mu(T=0) = 0$. Show that the result of part (a) above means that the chemical potential must remain at zero for all temperatures if particle number is to be conserved. (2 points)

(c) Using the results of (a) and (b) above, show that the average excitation energy, the change in the energy of the system from it s energy at $T = 0$ in three dimensions is given by:

$$\Delta E \equiv E(T) - E(0) = 4V \int \frac{d\vec{k}}{(2\pi)^3} E_+(\vec{k}) \frac{1}{1 + e^{\beta E_+(\vec{k})}}$$

(2 points)

(d) Evaluate the integral above for massless ($m = 0$) particles. (2 points)

(e) Calculate the heat capacity of such particles. (2 points)

$P = nf(\epsilon)$

a) means $nf(\mu+\delta) = n(1 - f(\mu-\delta))$

where $f_f(E) = \frac{1}{e^{\beta(E-\mu)}+1}$

$f(\mu+\delta) = \frac{1}{e^{\beta\delta}+1}$

$1-f(\mu-\delta) = 1 - \frac{1}{e^{-\beta\delta}+1} = \frac{e^{-\beta\delta}+1-1}{(e^{-\beta\delta}+1)}$

$= \frac{e^{-\beta\delta}}{e^{-\beta\delta}+1} = \frac{1}{1+e^{\beta\delta}} = f(\mu+\delta)$ ✓

b) $f(E_+) = 1 - f(E_-)$ to conserve particle #

and use $E_- = -E_+$

$\frac{1}{e^{\beta(E-\mu)}+1} = 1 - \frac{1}{e^{-\beta(E+\mu)}+1}$

$= \frac{e^{-\beta(E+\mu)}}{e^{-\beta(E+\mu)}+1} = \frac{1}{e^{\beta(E+\mu)}+1}$

For $\mu = -\mu$, $\mu = 0$ ✓

c) Use $\langle E \rangle = \int d^3x d^3k f(k) g(k) E_k$

$E(T) = \int d^3x d^3k f(k) g(k) [P(E_+)E_+ + (1-P(E_+))E_-]$ $\quad P(E_-) \equiv 1 - P(E_+)$

$-\ E(0) = \int d^3x d^3k f(k) g(k) (-E_+)$ $\quad$ ← all are in $E_- = -E_+$ state

$= \int d^3x d^3k f(k) g(k) [P(E_+)E_+ \mp E_+ + P(E_+)E_+ + E_+]$

$= \int etc\ [2P(E_+)E_+]$

$= 2\left[\frac{2V}{(2\pi)^3} \int d^3k\, E_+ \frac{1}{e^{\beta E_+}+1}\right]$ ✓

d) $E_+ = \hbar kc$

$\langle E \rangle = \frac{4V(4\pi)}{(2\pi)^3} \int \frac{k^2 dk\, \hbar kc}{e^{\beta\hbar kc}+1}$ use Schaum's $= \frac{V\pi}{6\beta^3\hbar^2c^2} = \langle E \rangle$

e) $C_V = \frac{\partial E}{\partial T} = -k\beta^2 \frac{\partial E}{\partial \beta}$ use E from d.

$= -k\beta^2\left(\frac{-V\pi 3}{6\beta^4\hbar^2c^2}\right) = \frac{3kV\pi}{6\beta^2\hbar^2c^2}$

$= \frac{k^3V\pi T^2}{2\hbar^2c^2}$

[1] Technically this means the total energy of the system diverges. If this bothers you, you can assume some large cut-off to the wavevectors, $\hbar k_{max}c >> kT$, which will have no effect on your final answers.

See Garrod 176-7 & HW 4, Q1 (not Bose though!)

6. Consider a set ($N >> 1$) of spinless bosons confined in a harmonic oscillator potential. The characteristic frequency of the harmonic potential is ω_0, and $\hbar\omega_0 << kT$, where T is the temperature and k is Boltzmann's constant.

   (a) Assuming the system is one dimensional, so that the energy of the system is given by $E = \hbar\omega_0(n + 1/2)$, calculate $N(T, V, \mu)$, in the above limit, where μ is the chemical potential. **(3 points)**

   (b) Show that there is no Bose-Einstein transition for this system in 1D. **(1 points)**

   (c) Assuming the system is two dimensional, calculate $N(T, V, \mu)$, again in the limit $\hbar\omega_0 << kT$. **(3 points)**

   (d) Show that there is a Bose-Einstein transtion and calculate the critical temperature as a function of the number of particles. (Do **not** simply quote a result.) **(3 points)**

a) $Q = \sum_{N=0}^{\infty} e^{\beta\mu N} \sum_{\{\sigma\}} e^{-\beta E_\sigma} = \sum_{N=0}^{\infty} \sum_{\{n_i\}} e^{\beta\mu \sum_i n_i - \beta E_i \sum_i n_i} = \prod_i \sum_{n_i=0}^{\infty} e^{-\beta(E_i-\mu)n_i} = \prod_i \frac{1}{1-e^{-\beta(E_i-\mu)}}$

$\mathcal{G} = G = -\frac{1}{\beta}\ln Q = +\frac{1}{\beta}\sum_i \ln\left(1-e^{-\beta(E_i-\mu)}\right)$ $\quad N = -\frac{\partial \mathcal{G}}{\partial\mu} = \frac{1}{\beta}\sum_i \frac{-1\left(-e^{-\beta(E_i-\mu)}\right)(\beta)}{1-e^{-\beta(E_i-\mu)}} = \sum_i \frac{1}{e^{\beta(E_i-\mu)}-1} = \sum_i f_B(E_i)$ ✓

Since $\hbar w \ll kT$, can integrate over n:

$N = \int \frac{1}{e^{\beta(\hbar w n + \frac{\hbar w}{2} - \mu)}-1}\,dn$ use $u = 1 - e^{-\beta(\hbar w n + \frac{\hbar w}{2} - \mu)}$ $\Big\}$ $N = -\frac{1}{\beta\hbar w}\ln\left(1-e^{-\beta(\frac{\hbar w}{2}-\mu)}\right)$

b) Tried to see limit of N as $T \to 0$, $\beta \to \infty$ but is terrible.
Can also find $\mu(N)$, set $\mu = \epsilon_f$ (ground state energy) and $T=0$ and see if N is restricted.

$\ln\left(1-e^{-\beta(\frac{\hbar w}{2}-\mu)}\right) \approx -e^{-\beta(\frac{\hbar w}{2}-\mu)}$ by the $\ln(1+x)$ expansion for small x. Seems a bit shady but.

$+N\beta\hbar w = e^{-\beta(\frac{\hbar w}{2}-\mu)}$

$-kT\ln N\beta\hbar w = \frac{\hbar w}{2} - \mu \rightarrow \mu = \frac{\hbar w}{2} + kT\ln N\beta\hbar w$ works for all N (no restrictions), so no condensate!

c) $N = \sum_{n_x}\sum_{n_y} \frac{1}{e^{\beta(\hbar w(n_x+n_y+1)-\mu)}-1}$ Set $n_x+n_y = N$ but have to multiply whole expression by degeneracy in N (all the ways n_x & n_y can make up $N = N+1$)

$= \sum_N \frac{N+1}{e^{\beta(\hbar w(N+1)-\mu)}-1} = \int_0^\infty \frac{(N+1)\,dN}{e^{\beta(\hbar w(N+1)-\mu)}-1} = \int_0^\infty \frac{u\,du}{e^{-\beta\mu}e^{\hbar w u\beta}-1} \approx \frac{1}{(\beta\hbar w)^2}\int_0^\infty \frac{u\,du}{z^{-1}e^{u}-1}$

$u = N+1$
$du = dN$

$N = \frac{g_2(e^{\beta\mu})}{(\beta\hbar w)^2}$

This is a trick, essentially the ground state didn't contribute. So we shd get a condensate.

d) Set $\mu = 0$ & $T = 0$. If we have a bound on N then condensate.

$N = \frac{g_2(\phi)}{(\hbar w)^2}(k)(0) = 0$ condensate! Set $\mu = 0$ solve for $T = T_c$: $\sqrt{\frac{N\hbar^2 w^2}{g_2(0)k^2}} = T_c = \frac{\hbar w\sqrt{N}}{k\sqrt{g_2(\phi)}}$

6. Consider a fictitious spin 5/2 fermion with the charge of an electron but with a dispersion relationship

$$E = v_0 p.$$

where $p \equiv |\vec{p}|$. We will call this particle the "offon." Assume that your offons are confined in a three dimensional sample and are non-ineracting. We will work in the Grand Canonical Ensemble.

(a) Determine the density, $\rho = \langle N \rangle / V$, as a function of the chemical potential μ (or the fugacity, $z \equiv e^{\beta\mu}$), T, and V. **(3 points)**

(b) What is the offonic Fermi energy (μ at T =0) as a function of their density? (*Hint:* This should not involve any complicated integrals). **(3 points)**

(c) Derive a series expansion in z for the grand canonical free entropy, $\Xi = \frac{PV}{kT} = \log \mathcal{Z}$, where $\mathcal{Z}$ is the grand canonical partition function. **(4 points)**

a) $Q = \sum_{N=0}^{\infty} e^{\beta\mu N} \sum_{\{\sigma\}} e^{-\beta E_\sigma} = \sum_{N=0}^{\infty} \sum_{\{n_i\}} e^{\sum_i n_i \beta\mu - \beta \sum_i n_i E_i} = \prod_i \left(1 + e^{-\beta(E_i - \mu)}\right)^6$ since $n_i = 0, 1$, but degeneracy of $2S+1$ per state

$\mathcal{G} = -\frac{1}{\beta} \ln Q = -\frac{6}{\beta} \sum_i \ln\left(1 + e^{-\beta(E_i-\mu)}\right)$ $N = -\frac{\partial \mathcal{G}}{\partial \mu} = \cancel{\frac{1}{\beta}} \sum_i \frac{6 e^{-\beta(E_i-\mu)}}{1 + e^{-\beta(E_i-\mu)}} (\cancel{\beta}) = \sum_i \frac{6}{e^{\beta(E_i-\mu)}+1}$ is definition for fermions, good.

Now $\sum_i$ turns into a "really big box", $\int \frac{d^3x\, d^3p}{(2\pi\hbar)^3}$

$N = \frac{6V(4\pi)}{(2\pi\hbar)^3} \int \frac{p^2 dp}{e^{\beta(v_0 p - \mu)}+1} = \frac{3V}{\pi^2\hbar^3} \int \frac{p^2 e^{-\beta(v_0 p-\mu)}}{1 - e^{-\beta(v_0 p - \mu)}} dp$ $\Bigg\}$ $\rho = \frac{\langle N \rangle}{V} = \underline{\underline{\frac{3}{\pi^2\hbar^3} \int \frac{p^2 dp}{e^{\beta(v_0 p-\mu)}+1}}} = \underline{\underline{\frac{3}{\pi^2\hbar^3} \int \frac{p^2 dp\, z e^{-\beta v_0 p}}{1 + z e^{-\beta v_0 p}}}}$

where $z = e^{\beta\mu}$

b) At $t=0$, $\mu = \epsilon_f$. $N = \int_0^\infty f(\epsilon) d\epsilon$ where $f(\epsilon) = \Theta(\epsilon - \epsilon_f)$. $\epsilon = v_0 p \to p^2 = \frac{\epsilon^2}{v_0^2}$

$d\epsilon = v_0 dp \to dp = \frac{d\epsilon}{v_0}$

$\rho = \frac{3}{\pi^2\hbar^3} \int_0^\infty \frac{\epsilon^2 d\epsilon}{v_0^3} \Theta(\epsilon - \epsilon_f) = \frac{3\epsilon_f^2}{\pi^2\hbar^3 v_0^3} \to \underline{\underline{\epsilon_f = \sqrt{\frac{\rho \pi^2 \hbar^3 v_0^3}{3}}}}$

c) $\psi = \ln Q = 6 \sum_i \underbrace{\ln\left(1 + e^{-\beta(E_i-\mu)}\right)}_{\text{assume small}} = \frac{6V}{(2\pi\hbar)^3} \int_0^\infty 4\pi p^2 dp \ln\left(1 + e^{-\beta(v_0 p - \mu)}\right) = \frac{3V}{\pi^2\hbar^3} \int_0^\infty \underbrace{\ln\left(1 + e^{-\beta(v_0 p-\mu)}\right)}_{\approx x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots} p^2 dp$

change to integral

$$\psi \cong \frac{3V}{\pi^2\hbar^3} \left[\int_0^\infty e^{-\beta(v_0 p-\mu)} p^2 dp - \frac{1}{2} \int e^{-2\beta(v_0 p - \mu)} p^2 dp + \cdots \right]$$

Problem 5 (10 Points):

See Schroeder p290ff.

The distribution function for an ideal Bose gas is given by,

$$f(\vec{x},\vec{p}) = g\ [\ e^{(\epsilon-\mu)/kT} - 1\ ]^{-1}$$

a. Define all the quantities found in $f(\vec{x},\vec{p})$. **(1 Points)**

b. What is the value of g for photons? **(1 Points)**

c. What is the meaning of the distribution function? Sketch the distribution as a function of energy. Make sure to label your sketch with the parameters. **(1 Points)**

d. For photons to be in thermal equilibrium there must be at least a small amount of matter present, since the interaction between photons is negligible. What processes bring the photons into equilibrium with the matter? **(1 Points)**

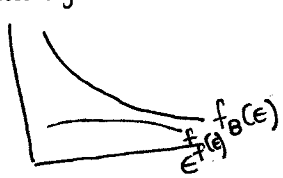
e. Use the information in part d. and the definition of chemical potential, $\mu = \partial F/\partial N|_{T,V}$ to explain why the chemical potential of photons must be zero. **(2 Points)**

f. Find the mean energy density of a photon gas in thermal equilibrium at temperature T. **(4 Points)**

a) g = degeneracy of each state; k = Boltzmann const.; T = temp. μ = chemical pot'l. ϵ = energy. $e^{\beta\mu}$ = fugacity
↳ $= 2S+1$ if spin

b) 2 - photons apparently have a polarization of 2. They cannot have spin 0 b/c they are massless. So it's $2S+1-1 = \underline{\underline{2}}$

c) Is the probability of that energy state being filled, or the probability of a particle having that energy state



d) Photoelectric effect, Compton scattering - these do not conserve N_γ. Radiation (blackbody).

e) Photons interact by other matter absorbing or creating them. So ex. $e \leftrightarrow e + \gamma$ but at equil, $\mu_i = \mu_f$ so $\mu_e = \mu_e + \mu_\gamma$. For this to work, μ_γ must be 0. Also we can see that $\frac{\partial F}{\partial N} = 0$ at equilibrium for photons, so that F doesn't change at equil, but N can change - see Schroeder p290.

f) $\langle E\rangle = \int E f(E)\, d^3E = \frac{V2(4\pi)}{h^3}\int \frac{p^2 dp\, pc}{e^{\beta(pc-\mu)}-1}$ ($\mu=0$) use $\left.\begin{matrix} E = pc \\ dE = c\,dp \end{matrix}\right\}$ the $\langle u\rangle$ is the full integrand divided by V $\rightarrow \underline{\underline{\langle u\rangle = \frac{8\pi V}{V h^3 c^3}\frac{\epsilon^3}{e^{\beta\epsilon}-1}}}$ changing the integration variable to $x = \beta\epsilon$ gives the familiar $u \sim T^4$

makes sense for a black body

S-06

Problem 6 (10 Points):

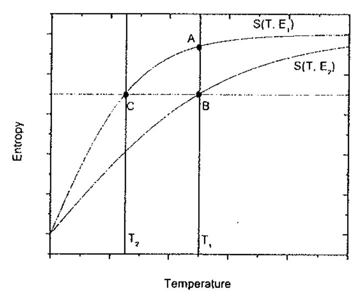
A sample consists of N independent electric dipoles. Each dipole has two possible quantum states with energies $\pm\mu E$ where E is the magnitude of an externally applied electric field. The lower energy state has dipole moment μ and the higher energy state has dipole moment $-\mu$.

a. Find the total electric dipole moment of the sample in an electric field E at temperature T. **(2 Points)**

b. What is the entropy of the sample? **(2 Points)**

c. Without using your result in b. explain physically what the entropy should be in the limits of $E \to 0$ and $E \to \infty$. **(2 Points)**

Entropy versus temperature curves for two values of electric field are shown below. Imagine that the sample is initially at state A, with temperature T_1 and field E_1.



d. How much heat must be extracted from the sample to move it from state A to state B, maintaining its temperature at T_1 while the field is raised from E_1 to E_2? **(2 Points)**

ate B, it is thermally isolated and the field to E_1, bringing the system from state B to erature of the sample once it reaches state riables given in the problem? **(2 Points)**

b) $F = E - TS$

d) $dQ = TdS$

$$Q = \int_{S_i}^{S_f} T\,dS = T\int_{S_i}^{S_f} dS = T(S_f - S_i)$$

$$= T_1(S(T_1, E_2) - S(T_1, E_1))$$

e) $dQ = 0$, is T_2

b. Let m_0 denote the difference between the fraction of atoms with $M_s = \frac{1}{2}$ and the fraction with $M_s = -\frac{1}{2}$; $m_0 = (N_+ - N_-)/N$. Derive the following approximate implicit equation for m_0 in the limit of zero field strength ($H \rightarrow 0$):

$$B_0 + fJm_0 = k_B T \tanh^{-1}(m_0)$$

(2 Points)

c. From the expression in b., derive an expression for the critical temperature T_c for spontaneous magnetization. Express your answer in terms of f, J, and Boltzmann's constant. **(2 Points)**

d. Derive the *value* of the critical exponent β (the degree of the coexistence curve) that describes how the order parameter $M_0(T)$ behaves as the temperature T approaches the critical temperature T_c from below:

$$M_0 \sim (\frac{T - T_c)}{T_c}) \text{ for } T \leq T_c$$

(3 Points)

Problem 6 (10 Points):

A sample consists ... Each dipole has two pos... ere E is the magnitude c... ower energy state has di... has dipole moment -μ.

$\mu = \frac{\partial F}{\partial N}$ $F = -\frac{1}{\beta}\ln Z$

n_1: $Z = \sum_{n=0}^{\infty} e^{-\beta E_n} = e^{-\beta\mu E} \rightarrow Z = (e^{-\beta\mu E})^{N_1}$

$F_1 = -\frac{N_1}{\beta}\ln(e^{-\beta\mu E}) = \frac{N}{\beta}\beta\mu E = N\mu E$

$\mu_1 = \mu E \frac{11}{0}$

a. Find the t... an electric field E at tem...

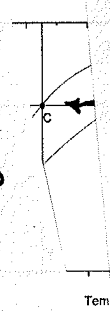
b. What is the ... S solve for S

c. Without usin... entropy should be in the ...

Entropy versu... ic field are shown below. ... at state A, with temperature T_1 a...

calc for one particle then at end mult by N since they're non interacting

ask Brian if get stuck



$<\mu> = \sum \mu_i P(\mu_i)$

$= \frac{1}{\beta}\frac{\partial}{\partial E}\ln Z$

E field

$<\mu> = \frac{1}{Z}\sum \mu_i e^{-\beta\mu_i E}$

$M = \frac{1}{\beta}\frac{\partial}{\partial B}\ln Z$

$\frac{1}{Z}\frac{\partial}{\partial B}Z = \frac{\partial}{\partial B}\ln Z$

often $H = -\mu B$

then $<\mu> = \frac{1}{Z}\sum_i \mu_i e^{-\beta E_i}$

$= \frac{1}{Z}\sum_i \mu_i e^{+\beta\mu B}$

so whether H is >0 or <0 determines whether your formula to get μ gets a (-) sign out front

a) $dQ = TdS$

$Q = \int_{S_i}^{S_f} TdS$

$Q = T\int_{S_i}^{S_f} dS$

$= T(S_2 - S_1)$

Problem 6 (10 Points):

Consider a white dwarf star that is composed of fully ionized ^{12}C and ^{16}O (a neutral plasma). The particle density of the star is uniform, and the electrons must be treated relativistically, E=pc.

a. Derive a relation between the Fermi energy of the electrons and the electron density. **(2 Points)**

b. Derive a relation between the average kinetic energy of the electrons and the Fermi energy. **(1 Points)**

c. The mass density is 10^{12}kg/m^3. Calculate the average kinetic energy of an electron, in MeV. (One MeV=1.6 x 10^{-13}J.) **(1 Points)**

d. The temperature is 10^9K. Calculate the average kinetic energy of the nuclei. **(1 Points)**

e. According to the virial theorem the internal energy of a system is approximately equal to its gravitational potential energy. For a sphere of uniform density, the gravitational potential energy is 3GM^2/5R. Derive an expression for the mass of the white dwarf in terms of fundamental constants only. **(3 Points)**

f. Calculate the mass of the white dwarf in solar masses. (1 solar mass = 2 x 10^{30}kg). A white dwarf in which the electrons are relativistic is unstable with respect to collapse, so the quantity that you have calculated is approximately the maximum mass of a white dwarf, a quantity called the Chandrasekhar mass (1.4 solar masses). Does your numerical result look reasonable? Why or why not? **(2 Points)**

a) $N = 2\left(\frac{4}{3}\pi p_f^3\right)\frac{V}{h^3}$ & $\epsilon_f = p_f c \rightarrow p_f = \frac{\epsilon_f}{c} \rightarrow N = \frac{8\pi\epsilon_f^3}{3c^3h^3}V \rightarrow \underline{\underline{n = \frac{N}{V} = \frac{8\pi\epsilon_f^3}{3c^3h^3}}} \rightarrow \epsilon_f = \left(\frac{3c^3h^3N}{8\pi V}\right)^{1/3}$

b) $E_{tot} = \int_0^N \epsilon_f \, dN = \frac{3^{1/3}ch}{2\pi^{1/3}L}\int_0^N N^{1/3}dN = \frac{3^{1/3}ch}{2\pi^{1/3}L}\,\frac{3}{4}N^{4/3}$ & $E_{avg} = \frac{E_{tot}}{N} = \underline{\underline{\frac{3}{4}\epsilon_f}}$

c) To get density of e⁻'s: (n_e) $\rho = 10^{12}\,kg/m^3 \rightarrow n_e = \frac{10^{12}}{0.014} \times (6.22\times10^{23}) \times 14 = 6.22\times10^{38}\, e's/m^3$ so $\underline{\underline{E_e \cong 3.9\,MeV}}$

d) $E_i = \frac{3}{2}kT = \underline{\underline{1.3\times10^{-2}\,MeV}}$ is much smaller.

e) $E_{int\,tot} = \frac{2\pi\epsilon_f^4}{c^3h^3} \cong \frac{3Gm^2}{5R} \rightarrow \underline{\underline{m \cong \sqrt{\frac{10\pi R\epsilon_f^4}{3Gc^3h^3}}}}$

Better: $\left(\frac{4}{3}\pi R^3\right)n_e\,\frac{3}{4}hc\left(\frac{3n_e}{8\pi}\right)^{1/3} = \frac{3}{5}\frac{Gm^2}{R}$

$M = \frac{8\pi}{3}R^3 n_e m_p$

f) $3.9\,MeV \cong \frac{3Gm^2}{5R} \rightarrow M = 8.5\times10^{30}\,kg = 4.1$ solar masses looks a bit heavy