Classical Mech Main Points

Qualifier

Winter 2012

1 Newtonian Mechanics

- Set $F_N = 0$ to find the point when two objects separate (ex. ball rolls off hemisphere)
- Momentum ($p = mv$, $L = I\omega$) is conserved for all collisions; energy is conserved for elastic collisions
- Force $= -\nabla U$
- For periodic motion, if the equation of motion is $\ddot{x} + \xi x = 0$, the frequency is $\omega = \sqrt{\xi}$. If the equation has a term linear in $\dot{x}$, that is a damping term.
- Power: $P = \frac{dE}{dt} = \frac{\Delta W}{\delta t} = \vec{F} \cdot \vec{v} = \vec{\tau} \cdot \vec{\omega}$

1.1 Angular Motion

- Use $v = \omega r$, $x = \theta r$, $a = \alpha r$ for basic angular motion
- Circular motion: $ma = \frac{mv^2}{r} = m\omega^2 r$
- **Torque**: $\frac{dL}{dt} = \tau = \vec{r} \times \vec{F} = I\alpha = Fd\sin\theta$
- Period $T = \frac{2\pi}{\omega}$
- Remember: it's often easier to find $d\sin\theta$ than to find d and θ separately
- To derive moment of inertia: $I = \int r^2 dm$; solve for dm in terms of dr
- Can still also use $\Sigma F = ma$ if it helps. Consider all forces acting at same point (point particle)
- **Orbits**: $\frac{\partial^2 V_{eff}}{\partial r^2} > 0$ for **stable orbits**. Use $\frac{\partial V}{\partial r} = 0$ for circular orbits
- **Parallel Axis Theorem**: $I_{new} = I_{original} + MR^2$

Helpful moments of inertia:

- **sphere**: $I = \frac{2}{5}MR^2$
- **disc**: $I = \frac{1}{2}MR^2$

Rocket Ships: Use m = mass of ship, dm'=ejected mass, v=velocity of ship, $-u$=ejected mass velocity relative to ship. Then we have:

$$p_i = p_f \rightarrow 0 = (m - dm')(v + dv) + dm'(v - u) \tag{1}$$

Set $v = 0$ for simplicity, and $dm = -dm'$. After that it's mostly algebra/calculus.

2 Virtual Work

The principle of virtual work presents an alternative to Newtonian solutions for force problems. This method uses the equations:

$$\delta W = \sum_i \vec{F}_i^a \cdot \delta \vec{r}_i = 0 \qquad \delta W = \sum_i Q_i^a \delta q_i = 0 \tag{2}$$

In these equations, $\vec{F}_i^a$ represent the net applied forces, and Q_i^a represent the differentiated constraint equations. Transform the Q_i^a equation into the generalized (simplest) coordinates, and solve the resulting equations.

For example, if the constraint equation is for two blocks connected by a massless rod: $x^2 + y^2 - l^2 = 0$, with $x = l\cos\theta$ and $y = l\sin\theta$:

$$\delta W = \sum_i Q_i^a \delta q_i = 0 \to 2x\delta x + 2y\delta y = 0 \to \delta x \cos\theta + \delta y \sin\theta = 0 \tag{3}$$

2.1 D'Alembert's Principle

The virtual work method given previously works for systems in static equilibrium. To generalize this method to dynamic systems, D'Alembert introduced a new "force of inertia" that modifies the virtual work equation that governs forces:

$$\delta W = \sum_i \left[\vec{F}_i^a - m_i \ddot{r}_i\right] \cdot \delta \vec{r}_i = 0 \tag{4}$$

3 Lagrangian & Hamiltonian

3.1 Lagrangian

- $L = T - U$
- Euler Lagrange Equation:
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0 \tag{5}$$
- We can always add a total time derivative of a function to the Lagrangian for free (without changing equations of motion):
$$L' = L + \frac{dF(q, \dot{q}, t}{dt} \tag{6}$$
This kind of trick can give a simplified Hamiltonian, even making it a constant of the motion.
- A variable q_i is **cyclic** if it does not appear in the Lagrangian. In that case, the associated momentum p_i is conserved/constant, and subtracting the associated $p_i \dot{q}_i$ transforms the Lagrangian into the Routhian:
$$p_i = \frac{\partial L}{\partial \dot{q}_i} = \alpha_i \quad \to \quad R = L - \alpha_i \dot{q}_i \tag{7}$$

3.2 Hamiltonian

- Legendre Transformation: $H = p\dot{q} - L$
- $p_q = \frac{\partial L}{\partial \dot{q}}$
- Hamilton's equations of motion: $\dot{p}_q = -\frac{\partial H}{\partial q}$ and $\dot{q} = \frac{\partial H}{\partial p_q}$
- Solve for q(t) using the E-L equation or Hamilton's equations of motion (take $\frac{d\dot{q}}{dt}$ and plug in for $\dot{p}_q$)
- We can see that H is conserved (thus representing the total energy) if $\frac{\partial H}{\partial t} = 0$ and if it includes no terms that depend *linearly* on a momentum variable (only quadratically).

- We can go farther, and write a momentum-space "Lagrangian", similar to how we did the first Legendre transform: $K(p, \dot{p}, t) = q_i \dot{p}_i + H(q, p, t)$
- $KE_{cylindrical} = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\phi}^2 + \dot{z}^2 \right)$
- $KE_{spherical} = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2 \right)$

3.3 Undetermined Multipliers

If we can't include some constraints when writing the Lagrangian, we have to take these constraints into account in the Euler-Lagrange equation as undetermined multipliers:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = Q_i^a + \sum_{j=1}^{m} \lambda_j a_{ji} \tag{8}$$

Each λ_j corresponds to each constraint equation f_j, and each a_{ji} corresponds to $\frac{\partial f_j}{\partial q_i}$.
Each Q_i^a corresponds to applied forces that cannot be written as part of the potential energy:

$$Q_i = \frac{\partial \vec{r}_j}{\partial q_i} \cdot \vec{F}_j \tag{9}$$

A constraint is *holonomic* if:

$$\frac{\partial}{\partial y} \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} \frac{\partial f}{\partial y} \tag{10}$$

3.4 Canonical Transformations

"Guess" the Q & P to transform into in order to make $\frac{\partial H}{\partial t} = 0$. Show canonical by $[Q, P]_{q,p} = 1$

Use existing p and q definitions to find generating functions:

$$p = \frac{\partial F_1(q, Q)}{\partial q} \qquad P = -\frac{\partial F_1(q, Q)}{\partial P} \tag{11}$$

$$p = \frac{\partial F_2(q, P)}{\partial q} \qquad Q = \frac{\partial F_2(q, P)}{\partial P} \tag{12}$$

$$q = -\frac{\partial F_3(Q, p)}{\partial p} \qquad P = -\frac{\partial F_3(Q, p)}{\partial Q} \tag{13}$$

$$q = -\frac{\partial F_4(p, P)}{\partial q} \qquad Q = \frac{\partial F_4(p, P)}{\partial P} \tag{14}$$

The generating function(s) result in a new Hamiltonian:

$$K(Q, P, t) = H(q, p, t) + \frac{\partial F_2}{\partial t} \tag{15}$$

The new Hamiltonian results in corresponding new equations of motion:

$$\dot{P} = -\frac{\partial K}{\partial Q} \qquad \dot{Q} = \frac{\partial K}{\partial P} \tag{16}$$

$H = T + U$ if $\frac{\partial H}{\partial t} = 0$, no explicit time dependence, AND no terms linear in momentum/velocity

3.5 Small Oscillations with Effective Potentials

To find frequency of small oscillations:

1. Write the Hamiltonian and find the effective potential, V_{eff} (all terms that depend on q)
2. Find $\frac{\partial^2 V_{eff}}{\partial q^2}|_{q=qmin}$ where q represents the variable with small oscillations
3. Write the V matrix as:
$$V = \frac{1}{2}\tilde{V}q^2 = \frac{1}{2}\frac{\partial^2 V_{eff}}{\partial q^2}|_{qmin}q^2 \tag{17}$$
4. Write the T matrix as:
$$T = \frac{1}{2}\tilde{T}\dot{q}^2 \tag{18}$$
5. Solve for the frequency using $\tilde{V}$ and $\tilde{T}$:
$$\tilde{V} - \omega^2\tilde{T} = 0 \tag{19}$$

Quick way to get frequency: Make the Lagrangian look like: $L = \frac{1}{2}m'\dot{\eta}^2 - \frac{1}{2}k'\eta^2$. Then $\omega = \sqrt{\frac{k'}{m'}}$

3.6 Variational Calculus

The Euler-Lagrange equation can also solve other physics of path minimization, such as the brachistone problem of minimizing time for a particle in a force field to travel between two points. To use the E-L for this type of problem:

1. Write an equation that describes the motion and the element to minimize, such as $dt = \frac{ds}{v}$. The element to minimize should be alone on the LHS.
2. Add an integration symbol on both sides: $t = \int \frac{ds}{v}$
3. Write the RHS differential in terms of path variables, such as dx and dy, in order to evaluate the integral, such as: $t = \int \frac{\sqrt{1+x'^2}}{\sqrt{2gy}}dy$
4. Use the E-L equation on the integrand, using the appropriate variables, such as: $\frac{\partial F}{\partial x} - \frac{d}{dy}\frac{\partial F}{\partial x'} = 0$
5. Solve the resulting equation by separation of variables, such as $x(y) = \int \sqrt{\frac{y}{(c^2/2g)-y}}dy$

4 Vector Potentials

Remember that the vector potential due to a particle in a magnetic field is:

$$\vec{A} = -\frac{1}{2}B_0(y\hat{x} - x\hat{y}) \tag{20}$$

And to find the potential, use:

$$U = q\phi - q\vec{A}\cdot\vec{v} \tag{21}$$

where ϕ represents the electric potential.

5 Small Oscillations

Standard coordinates define how the blocks are displaced *relative to each other*, while small coordinates (usually η) define how the blocks are displaced *relative to their original equilibrium position.* Start by writing the Lagrangian in standard coordinates, then transform to small coordinates. Then use these notations:

$$L = \frac{1}{2}\mathbf{T}\dot{\eta}_i\dot{\eta}_j - \frac{1}{2}\mathbf{V}\eta_i\eta_j \tag{22}$$

Use $\frac{\partial V}{\partial q_i}|_{q0i} = 0$ to find the minimum point q_{0i}, and $\mathbf{V} = \frac{\partial^2 V_{eff}}{\partial q_i^2}|_{q0i} = \frac{\partial^2 V_{eff}}{\partial \eta_i \eta_j}|_0$ to find $\mathbf{V}$.

Then use $\mathbf{T}$ and $\mathbf{V}$ to solve for the frequency(s):

$$|\mathbf{V} - \lambda\mathbf{T}| = 0 \tag{23}$$

where $\lambda = \omega^2$, to solve for the frequencies ω_i. To find the eigenvectors:

$$(\mathbf{V} - \lambda_i\mathbf{T})\,\vec{c}_i = 0 \tag{24}$$

these $\vec{c}_i$ also make up the amplitude ratios for λ_i, $\frac{A_1}{A_2}$:

$$(\mathbf{V} - \lambda_i\mathbf{T})\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0 \tag{25}$$

To normalize the eigenvectors:

$$\vec{C}_i = N_i\vec{c}_i \rightarrow \vec{C}_i^T\mathbf{T}\vec{C}_i = 1 \tag{26}$$

Solve for N_i. Finally, to write the displacement of the system as a function of time:

$$A_i = \vec{C}_i^T\mathbf{T}\eta(0) \tag{27}$$

$$\omega_i^2 > 0 \rightarrow \omega_i B_i = \vec{C}_i^T\mathbf{T}\dot{\eta}(0) \tag{28}$$

$$\omega_i = 0 \rightarrow B_i = \vec{C}_i^T\mathbf{T}\dot{\eta}(0) \tag{29}$$

The general solution can now be written as:

$$\vec{\eta}(t) = \sum_{\omega_i^2>0}\vec{C}_i(A_i cos\omega_i t + B_i\sin\omega_i t) + \sum_{\omega_i^2=0}\vec{C}_i(A_i + B_i t) \tag{30}$$

Smaller ω's correspond to more symmetry in the oscillation mode.

6 Central Forces & the Hamilton Jacobi Equation

Whenever we have two masses exerting a force on each other, we can move into the center of mass reference frame and consider the reduced mass combination acted on by a central force, since the center of mass of the system does not move.

Orbits & Stability

- A circular orbit is stable if $\frac{\partial^2 V_{eff}}{\partial r^2} > 0$
- To find the radius for circular orbit, set $\frac{\partial V}{\partial r} = 0$ and solve for r (can also use Hamilton's equations)
- To find the condition on the radius for circular orbit, find $\frac{\partial^2 V_{eff}}{\partial r^2} > 0$ and substitute in the radius for circular orbit

Steps for Solving Motion with the Hamilton-Jacobi

1. *Background*: We can transform H without loss of generality to $K = H + \frac{\partial S}{\partial t} = 0$. Assuming then that S, Hamilton's principle/generating function is separable ($S(q,t) = S_1(t) + S_2(q)$) and $p = \frac{\partial S}{\partial q}$, we can rearrange K to be:
$$\frac{1}{2m}\left(\frac{\partial S_2}{\partial q}\right)^2 + V(q) = -\frac{\partial S_1}{\partial t} \tag{31}$$
Now the variables are separated, and we can set both sides equal to a constant, E. This makes solving for S_1 and S_2 a matter of maths.

2. Write Hamilton's equation, and substitute $\frac{\partial S_2}{\partial q}$ for each p_q term. (S_2 is sometimes referred to as W)

3. Separate variables - this usually entails writing everything not dependent on r inside a bracket, and setting that bracket equal to α_3. (This is usually the total angular momentum, which we can see is a constant of the motion by finding $[L, H] = 0$). Or solve so that r is on one side of the equation, and θ and ϕ are on the other side, then set both sides equal to α_3.

4. Assuming W is separable (example $W(r,\theta,\phi) = W_r + W_\theta + W_\phi$), find integrals defining each component of W.

5. Use $p_q = \frac{\partial W}{\partial q}$ to find the meaning of α_2 and α_3.

6. Use the form $\frac{\partial W}{\partial E} = t + \beta$ to solve for the motion of r depending on E and α's.

7. *Additional*: It may be useful to also remember that $Q = \frac{\partial S_2}{\partial P} = \frac{\partial S_2}{\partial E}$ and $\dot{Q} = \frac{\partial H}{\partial P} = \frac{\partial H}{\partial E}$.

The "action", J is equivalent to $S_2(q)$ as long as $S(q,t)$ is separable:
$$J = \int p dq = \int P dQ \tag{32}$$
Given this J, the frequency of motion is:
$$\nu_i = \frac{\partial E}{\partial J_i} \tag{33}$$
where E came from integrating the action J and solving for $E(J)$.

7 The Poisson Bracket

The poisson bracket is a good method of determining which elements associated with a Hamiltonian are constants of motion:
$$\frac{du}{dt} = [u, H]_{qi,pi} + \frac{\partial u}{\partial t} \tag{34}$$
$$[u, H]_{qi,pi} = \sum_i^n \left(\frac{\partial u}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial H}{\partial q_i}\right) \tag{35}$$
For example, given angular momentum $J = q_1 p_2 - q_2 p_1$, the poisson bracket of J with H quickly shows that the angular momentum is conserved:
$$\frac{dJ}{dt} = [J, H]_{qi,pi} = 0 \tag{36}$$
In general, to find whether an element is a constant of motion:

1. Write the element A in terms of q_i and p_i

2. Write the Hamiltonian according to the physical description

3. Find $\frac{dA}{dt} = [A, H]_{qi,pi} + \frac{\partial A}{\partial t}$

For canonical variables:
$$[q_i, q_j] = 0 \quad [q_i, p_j] = \delta_{ij} \quad [p_i, p_j] = 0 \tag{37}$$
The poisson bracket also helps verify that transformations are properly canonical:
$$[Q, P]_{q,p} = 1 \tag{38}$$

8 Extra

8.1 Conservative Forces

A force is conservative if $\vec{\nabla} \times \vec{F} = 0$. In Cartesian coordinates, can find this as: $\frac{\partial F_i}{\partial j} = \frac{\partial F_j}{\partial i}$

8.2 Nonhomogeneous Equations

Solving a non-homogeneous equation requires the combination of a *particular* and a *complementary* solution:

$$\dot{y} + ay = b \qquad \rightarrow \qquad y(t) = y_p(t) + y_c(t) \tag{39}$$

1. The particular solution should be of the form $y_p(t) = At^2 + Bt + C$, keeping only the terms so that $y_p(t)$ is a polynomial of the same order as the right hand side of the original equation. So in this example, $y_p(t) = C$.
2. The complementary solution solves $y(t)$ for the right hand side equalling zero: $\dot{y} + ay = 0$. Solve this the usual way, including the constant of integration.
3. Write $y(t) = y_p(t) + y_c(t)$, and substitute these results back into the original equation. Use the original equation and initial conditions to solve for the constants of integration.

Remember that a second derivative equation of motion can be handled as a first derivative equation by writing it in terms of velocity instead of position: $\ddot{y} + a\dot{y} = b \rightarrow \dot{v}_y + av_y = b$

9 Coordinate Systems

9.1 Cartesian

Convert to spherical: $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

Convert to cylindrical: $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$

9.2 Spherical

$$\hat{r} = \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z} \tag{40}$$

$$\hat{\theta} = \frac{\partial\hat{r}}{\partial\theta} \ \& \ \hat{\phi} = \frac{\partial\hat{r}}{\partial\phi} \tag{41}$$

Derivation of a small chunk of circular area (such as in Kepler's law for orbits):

$$S = r\theta \rightarrow dS = rd\theta \rightarrow dA = R^2 d\theta \tag{42}$$

9.3 Cylindrical

$$\hat{r} = \cos\theta\hat{x} + \sin\theta\hat{y} \tag{43}$$

$$\hat{\theta} = \frac{\partial\hat{r}}{\partial\theta} \tag{44}$$