Electrodynamics Main Points

Qualifier

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1 Definitions

The position vector:

$$\vec{r} \equiv x\hat{x} + y\hat{y} + z\hat{z} \rightarrow \hat{r} = \frac{\vec{r}}{r} \tag{1}$$

The **source point** uses $\vec{r'}$ while the **field point** uses $\vec{r}$, so that [cursive r]=$\vec{r} - \vec{r'}$.

2 Electrostatics

2.1 Source Charges

Coulomb's Law (SI units):

$$\vec{F} = \frac{qQ}{4\pi\epsilon_0} \frac{1}{|\vec{r} - \vec{r'}|^2} \widehat{\vec{r} - \vec{r'}} \tag{2}$$

Electric field due to source charges:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \frac{q_i}{|\vec{r} - \vec{r'}|^2} \widehat{\vec{r} - \vec{r'}} \tag{3}$$

Such that $\vec{F} = Q\vec{E}$.

For continuous charge distributions:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{1}{|\vec{r} - \vec{r'}|^2} \widehat{\vec{r} - \vec{r'}} dq \tag{4}$$

The charge element dq takes different forms depending on how the charge is spread out:

$$dq \rightarrow \lambda dl' \rightarrow \sigma da' \rightarrow \rho d\tau' \tag{5}$$

Note that the unit vector $\widehat{\vec{r} - \vec{r'}}$ is not constant and cannot be taken out of the integral. In order to take the unit vector outside the integral, it must first be converted to Cartesian components, even if we then use curvilinear coordinates to perform the integration.

2.2 Flux

Flux through a surface S is measured as:

$$\Phi = \int_S \vec{E} \cdot d\vec{a} \tag{6}$$

This leads to **Gauss' Law**:

$$\int_S \vec{E} \cdot d\vec{a} = \frac{q_{enc}}{\epsilon_0} \rightarrow \nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \tag{7}$$

2.3 Electric Potential

To prove Maxwell's equation that no static electric fields have curl, simply use Stokes' theorem to show that:

$$\oint \vec{E} \cdot d\vec{l} = 0 \rightarrow \nabla \times \vec{E} = 0 \tag{8}$$

This leads to the definition of **electric potential**:

$$\vec{E} = -\nabla\Phi \rightarrow \Phi(\vec{r}) = -\int_0^{\vec{r}} \vec{E} \cdot d\vec{l} \tag{9}$$

Electric potential obeys superposition just as electric field does.

It is sometimes easier to calculate the potential first and then find the gradient to get the electric field. Assuming the reference point at infinity:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r'})}{|\vec{r} - \vec{r'}|} d\tau' \tag{10}$$

In this way we avoid the messy calculation of $(\widehat{\vec{r} - \vec{r'}})$.

2.3.1 Some Helpful Potentials

The potential of a dipole:

$$\Phi_{dip} = \frac{\vec{p} \cdot \vec{r}}{4\pi\epsilon_0 r^3} \tag{11}$$

2.4 Surface Charge Density

$$\sigma = -\epsilon\vec{E}|_{boundary} = -\epsilon\frac{\partial\Phi}{\partial n}|_{boundary} \tag{12}$$

3 Special Techniques

3.1 Separation of Variables

Remember that separable solutions to the LaPlace equation can simplify geometries. For example:

$$V(x, y, z) = V_x V_y V_z \rightarrow \frac{1}{V_x}\frac{\partial V_x}{\partial x} + \frac{1}{V_y}\frac{\partial V_y}{\partial y} + \frac{1}{V_z}\frac{\partial V_z}{\partial z} = 0 \tag{13}$$

If the coefficient is chosen **positive**, such as $\frac{1}{V_x}\frac{\partial V_x}{\partial x} = k^2$, then the solution is of the form:

$$V_x = Ae^{kx} + Be^{-kx} \tag{14}$$

If the coefficient is chosen **negative**, such as $\frac{1}{V_x}\frac{\partial V_x}{\partial x} = -k^2$, then the solution is of the form:

$$V_x = A\sin kx + b\cos ky \tag{15}$$

Usually choose the positive direction to be the direction in which nonzero potential lies. Another helpful identity to use in simplifying these integrals:

$$\int_0^a \sin(n\pi y/a)\sin(n'\pi y/a)dy = \frac{a}{2}\delta_{nn'} \tag{16}$$

4 Legendre Polynomials

A number of helpful identities and definitions can simplify some complicated geometries, especially if they have azimuthal symmetry:

$$\frac{1}{|\vec{r} - \vec{r}'|} = \sum_{l=0}^{l=\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma) \tag{17}$$

where γ is the angle between $\vec{r}$ and $\vec{r}'$.

Also in spherical dimensions, the potential that solves the Laplace equation $\nabla^2\Phi$ can be written as:

$$\Phi = \sum_{l=0}^{\infty}(A_l r^l + \frac{B_l}{r^{l+1}})P_l(\cos\theta) \tag{18}$$

This can then be simplified with boundary conditions.

Also the following identity can be helpful:

$$\int_{-1}^{1} P_l(x)P_{l'}(x)dx = \frac{2}{2l+1}\delta_{ll'} \tag{19}$$

5 Helpful Maths

5.1 Tensors and Vectors

Remember that the vector product is distributive:

$$\vec{A} \times (\vec{B} + \vec{C}) = (\vec{A} \times \vec{B}) + (\vec{A} \times \vec{C}) \tag{20}$$

The scalar triple product is the volume of the parallelepiped generated by the three vectors:

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B}) \tag{21}$$

Note that the order of the vectors is preserved. In this case $|\vec{B} \times \vec{C}|$ is the area of the base of the parallelepiped, and $|vecA\cos\theta|$ is the height.

The vector triple product follows the **BAC-CAB** rule:

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B}) \tag{22}$$

The gradient points in the direction of maximum increase of the function (magnitude of gradient gives slope):

$$\nabla T \equiv \frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z} \tag{23}$$

The divergence measures how much the vector spreads out from the point in question:

$$\nabla \times \vec{T} = \frac{\partial T}{\partial x} + \frac{\partial T}{\partial y} + \frac{\partial T}{\partial z} \tag{24}$$

The curl measures how much the vector curls around the point in question: $\nabla \times \vec{T}$. These operations result in *six* product rules:

$$\nabla(fg) = f\nabla g + g\nabla f \tag{25}$$

$$\nabla(\vec{A} \cdot \vec{B}) = \vec{A} \times (\nabla \times \vec{B}) + \vec{B} \times (\nabla \times \vec{A}) + (\vec{A} \cdot \nabla)\vec{B} + (\vec{B} \cdot \nabla)\vec{A} \tag{26}$$

$$\nabla \cdot (f\vec{A}) = f(\nabla \cdot \vec{A}) + \vec{A} \cdot (\nabla f) \tag{27}$$

$$\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B}) \tag{28}$$

$$\nabla \times (f\vec{A}) = f(\nabla \times \vec{A}) - \vec{A} \times (\nabla f) \tag{29}$$

$$\nabla \times (\vec{A} \times \vec{B}) = (\vec{B} \cdot \nabla)\vec{A} - (\vec{A} \cdot \nabla)\vec{B} + \vec{A}(\nabla \cdot \vec{B}) - \vec{B}(\nabla \cdot \vec{A}) \tag{30}$$

Remember that *the divergence of a curl and the curl of a gradient are always zero.*

The Laplacian comes up often: $\nabla \cdot (\nabla T) = \nabla^2 T$

The **fundamental theorem for gradients**:

$$\int_a^b (\nabla T) \cdot d\vec{l} = T(\vec{b}) - T(\vec{a}) \tag{31}$$

That is, the line integral of the gradient is path independent.

The **fundamental theorem for divergences** - Green's theorem:

$$\int_V (\nabla \cdot \vec{v}) d\tau = \int_S \vec{v} \cdot d\vec{a} \tag{32}$$

The **fundamental theorem for curls** - Stokes' theorem:

$$\int_S (\nabla \times \vec{v}) \cdot d\vec{a} = \int_P \vec{v} \cdot d\vec{l} \tag{33}$$

5.2 Curvilinear Coordinates

5.2.1 Spherical Polar Coordinates

Converting from Cartesian:

$$x = r \sin\theta \cos\phi \qquad y = r \sin\theta \sin\phi \qquad z = r\cos\theta \tag{34}$$

Likewise,

$$\hat{r} = \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z} \tag{35}$$

$$\hat{\theta} = \cos\theta\cos\phi\hat{x} + \cos\theta\sin\phi\hat{y} - \sin\theta\hat{z} \tag{36}$$

$$\hat{\phi} = -\sin\phi\hat{x} + \cos\phi\hat{y} \tag{37}$$

Note that since the direction of the spherical components change relative to the cartesian coordinates, straightforward addition of two vectors by spherical component is usually not possible.

The length element:

$$d\vec{l} = dr\hat{r} + rd\theta\hat{\theta} + r\sin\theta d\phi\hat{\phi} \tag{38}$$

The area element:

$$da = r^2 \sin\theta d\theta d\phi \tag{39}$$

The volume element:

$$d\tau = r^2 \sin\theta dr d\theta d\phi \tag{40}$$

The gradient:

$$\nabla T = \frac{\partial T}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\phi} \tag{41}$$

The divergence:

(42)

The curl:

(43)

5.2.2 Cylindrical Coordinates

Converting from Cartesian:

$$x = r\cos\phi \qquad y = r\sin\phi \qquad z = z \tag{44}$$

Likewise,

$$\hat{r} = \cos\phi\hat{x} + \sin\phi\hat{y} \tag{45}$$

$$\hat{\phi} = -\sin\phi\hat{x} + \cos\phi\hat{y} \tag{46}$$

$$\hat{z} = \hat{z} \tag{47}$$

The length element:

$$d\vec{l} = dr\hat{r} + rd\phi\hat{\phi} + dz\hat{z} \tag{48}$$

The volume element:

$$d\tau = rdrd\phi dz \tag{49}$$

The gradient:

$$\nabla T = \frac{\partial T}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial T}{\partial \phi}\hat{\phi} + \frac{\partial T}{\partial z}\hat{z} \tag{50}$$

The divergence:

(51)

The curl:

(52)

The Laplacian:

(53)

5.3 The Dirac Delta Function

5.3.1 The 1D Case

$$\int_{-\infty}^{\infty} \delta(x)dx = 1 \tag{54}$$

The delta function picks out $x = 0$ (or x' as the case may be):

$$\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0) \tag{55}$$

5.3.2 The 3D Case

The delta function can be generalized to three dimensions:

$$\delta^3(\vec{r}) = \delta(x)\delta(y)\delta(z) \tag{56}$$

5.4 Theory of Vector Fields

Theorem 1: Curl-less/"Irrotational" Fields: If a vector satisfies one of the following, it satisfies all of them.

- $\nabla \times \vec{F} = 0$ everywhere
- $\int_a^b \vec{F} \cdot d\vec{l}$ is independent of path for any given end points
- $\int \vec{F} \cdot d\vec{l} = 0$ for any closed loop
- $\vec{F}$ is the gradient of a scalar, $\vec{F} = -\nabla V$

Theorem 2: Divergence-less/"Solenoidal" Fields: If a vector satisfies one of the following, it satisfies all of them.

- $\nabla \cdot \vec{F} = 0$ everywhere

- $\int \vec{F} \cdot d\vec{a}$ is independent of surface for any given boundary line
- $\int \vec{F} \cdot d\vec{a} = 0$ for any closed surface
- $\vec{F}$ is the curl of some vector, $\vec{F} = \nabla \times \vec{A}$